ANALYSIS OF DISCRETE NONLINEAR SYSTEMS BY TRANSFORM METHODS

A thesis submitted to the Faculty of
Engineering of the University of Aston
in Birmingham for the degree of,
Doctor of Philosophy

by

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PREFACE

This thesis describes research carried out at the Electrical Engineering Department of the University of Aston in Birmingham from 1971 to 1974. No part of this thesis has been submitted for a qualification at another university, but part of the contents of chapters 2 and 3 have appeared in the following publication:

Barker,H.A. and Ambati,S.: 'Nonlinear sampled-data system analysis by multidimensional z transforms', Proc. IEE., Vol. 119, 1972, pp. 1407-1413.

I wish to express my sincere gratitude to my supervisor, Professor H.A. Barker for providing invaluable help and guidance. I also wish to express my sincere thanks to the University of Aston in Birmingham for providing the financial assistance throughout the period of my work, to the Institution of Electrical Engineers for awarding, jointly with Professor H.A. Barker, the Institute's Heaviside premium for the above paper, and to Professor J.E. Flood, Head of the Department of the Electrical Engineering, for the provision of facilities without which this research work could not have been completed.

Sambati

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SUMMARY

This thesis is concerned with the analysis, modelling and simulation of nonlinear systems by transform methods. The systems are characterised by Volterra functional series, and a sequential process relating the n-dimensional z transform of a n^{th}-order Volterra kernel to n-dimensional Laplace transform of the kernel is described.

For nonlinear systems cascaded with a data-hold device, it is shown that the multidimensional z transfrom of the cascade combination may be obtained by applying the process to a derived multidimensional Laplace transform. The analysis of asynchronous sampled-data systems is investigated by multidimensional modified z transformation. The operator algebra is extended for the analysis of nonlinear sampled-data feedback systems.

A systematic procedure is developed for the synthesis of a discrete Volterra kernel using finite number of multipliers and first and second-order linear discrete systems. This procedure is used to obtain a discrete simulator for a continuous nonlinear system preceded by a data-hold device and a method to reduce the number of multipliers is suggested.

The state space characterisation of multivariable systems with multiplicative or functional nonlinearities is investigated. It is shown that the multidimensional Laplace transform kernels characterising such a system may be determined and synthesised by establishing an explicit input-output relationship in terms of the state transition matrix. The state variable analysis of nonlinear multivariable sampled-data system associated with data-hold devices is also investigated.

The Volterra series and the transform methods are applied to a practical feedback demodulator, and its nonlinear model derived. The spectral results obtained by digital simulation of the model, are

compared with those obtained by others in order to establish the superiority of this method in its generality and in achieving better results.

It is concluded that the method of deriving the multidimensional z transform of a Volterra kernel is extremely useful as it allows the continuous nonlinear system to be simulated on a digital computer, which is often required in dynamic system investigation.

List Of Principal Symbols

t, τ = time variables

$f_n(t_1,t_2,\ldots,t_n)$ = function of n time variables

M.D.L.T = multidimensional Laplace transform

M.D.Z.T = multidimensional z transform

M.D.M.Z.T = multidimensional modified z transform

M.D.I.L.T = multidimensional inverse Laplace transform

M.D.I.Z.T = multidimensional inverse z transformation

M.D.I.M.Z.T = multidimensional inverse modified z transformation

I.L.T = inverse Laplace transformation

I.Z.T = inverse z transformation

I.M.Z.T = inverse modified z transformation

S.P = sequential process

A.V.M.D.L.T = association of variables in M.D.L.T

A.V.M.D.Z.T = association of variables in M.D.Z.T

A.V.M.D.M.Z.T = association of variables in M.D.M.Z.T

s = Laplace transform variable

T = sampling period

z = z transform variable

m = modified z transform variable, $0 \leq m < 1$

$F_n(s_1,s_2,\ldots,s_n)$ = M.D.L.T of $f_n(t_1,t_2,\ldots,t_n)$

$F_n(z_1,z_2,\ldots,z_n)$ = M.D.Z.T of $f_n(t_1,t_2,\ldots,t_n)$

$F_n(m,z_1,z_2,\ldots,z_n)$ = M.D.M.Z.T of $f_n(t_1,t_2,\ldots,t_n)$

${}^{L}_{Z}F^{n-r}_{r}(z_1,z_2,\ldots,z_r,s_{r+1},\ldots,s_n)$ = multidimensional mixed z and Laplace transform of $f_n(t_1,t_2,\ldots,t_n)$

${}^{L}_{MZ}F^{n-r}_{r}(m,z_1,z_2,\ldots,z_r,s_{r+1},\ldots,s_n)$ = multidimensional mixed modified z and Laplace transform of $f_n(t_1,t_2,\ldots,t_n)$

$u(t)$ = nonlinear system input

$*$ = denotes sampling of a continuous-time function

$U_1(s)$ = Laplace transform of $u(t)$

$u^*(t) = u(iT)$ = nonlinear system sampled-data input

$y(t)$ = nonlinear system output

$w_1(\tau)$ = weighting function of linear system

$w_n(\tau_1, \tau_2, \ldots, \tau_n)$ = n^{th}-order kernel of nonlinear system

$y_n(t)$ = system output due to n^{th}-order kernel

$y_n(t_1, t_2, \ldots, t_n)$ = function equal to $y_n(t)$ when all arguments equal t.

$U_1(z)$ = z transform of $u^*(t)$

$Y_1(z)$ = associated z transform of $Y_n(z_1, z_2, \ldots, z_n)$

$Y_1(m,z)$ = associated modified z transform of $Y_n(m, z_1, z_2, \ldots, z_n)$

$\delta(t)$ = unit impulse function

$y_n(iT)$ = response of n^{th}-order kernel at sampled instants

$[y(iT)]_+$ = sampled response for positive step input

$[y(iT)]_-$ = sampled response for negative step input

$y_n(\langle i+m \rangle T)$ = intersampled response of n^{th}-order kernel, for $0 \leq m < 1$

c_r = real number in s_r plane

C_r = closed contour in z_r plane

$|z_r|$ = magnitude of z in z_r plane

$*$ = denotes complex convolution

$+$ = denotes addition of two signals or systems

$\cdot$ = denotes multiplication of two signals or systems

$\otimes$ = denotes linear system operation

$\odot$ = denotes inverse linear system operation

$\circledast$ = denotes nonlinear system operation

$\circledast$ = denotes cascading in continuous-time domain

r^*, s^*, y^* = sampled signals in the system, each of which has a generalised discrete Volterra series expansion

R, S, Y = z transforms of r^*, s^*, y^*, respectively

J,K,L,M,N = continuous nonlinear subsystems represented by a set of Volterra kernels

J^*, K^*, L^*, M^*, N^* = discrete equivalent of continuous systems J,K,L,M,N

$\mathcal{Z}$ = z transform of

$\mathcal{Z}^{-1}$ = inverse z transform of

L^{-1}	= inverse Laplace transform of
Im, Re	= imaginary part of; real part of
${}_{p}x(t)$	= state vector of dimension P x 1
${}_{p}\dot{x}(t)$	= time-derivative of state vector
${}_{r}u(t)$	= input vector of dimension R x 1
${}_{q}y(t)$	= output vector of dimension Q x 1
A,B,C,D,E,F,G	= multidimensional arrays
I	= identity matrix
${}_{p}x(0)$	= initial condition vector of dimension P x 1
'- - - - '	= denotes matrix partitioning
$\phi(t)$	= state transition matrix
$\Phi(s)$	= Laplace transform of $\phi(t)$
${}_{q}W_{nLMN\ldots K}(s_1,s_2,\ldots,s_n)$	= M.D.L.T of n^{th}-order kernel characterising the q^{th} output of an R-input, Q-output nonlinear system
${}_{p}x(kT)$	= discrete state vector of dimension P x 1, representing the states of the system at the sampling instants
${}_{r}u(kT)$	= sampled-data input vector of dimension R x 1
${}_{q}y(kT)$	= sampled-data output vector of dimension Q x 1
$\phi(kT)$	= discrete state transition matrix
$\Phi(z)$	= z transform of $\phi(kT)$
${}_{q}P_{nLMN\ldots K}(z_1,z_2,\ldots,z_n)$	= M.D.Z.T of n^{th}-order kernel characterising the q^{th} output of an R-input, Q-output nonlinear sampled-data system cascaded with zero-order hold
FM	= frequency modulation
FBFM	= feedback FM
VCO	= voltage controlled oscillator
FMD	= FM discriminator
IF	= intermediate frequency
RF	= radio frequency
AM/PM	= amplitude-to-phase modulation
ln	= natural logarithm

CONTENTS

CHAPTER 1

INTRODUCTION

1.1 Nonlinear System Characterisation

In practice, no physical system is linear over a wide operating range and it is well known that the development of a universal method for solving all nonlinear problems is virtually impossible because of the different nature of various nonlinear physical systems. However, there are mainly two methods of analysis, by which the input-output relation of a given nonlinear system may be characterised:(a) a differential equation (for continuous system) or a difference equation (for discrete system) approach and (b) a functional approach. The differential or difference equation approach is not general and gives little insight into the behaviour of systems other than the particular system being analysed. Further, in the differential or difference equation representation, the system output is not an explicit function of its input. On the other hand, the functional methods, which give an explicit input-output relationship, are general and apply to a large class of nonlinear systems.

In the functional representation, the input-output relationship of a time-invariant continuous nonlinear system is given by

$$\begin{aligned} y(t) &= \int_{-\infty}^{\infty} w_1(\tau)u(t-\tau)\, d\tau + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_2(\tau_1,\tau_2)u(t-\tau_1)u(t-\tau_2)\, d\tau_1 d\tau_2 \\ &\quad + \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} w_3(\tau_1,\tau_2,\tau_3) \prod_{r=1}^{3} u(t-\tau_r)\, d\tau_r + \cdots\cdots \\ &= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w_n(\tau_1,\tau_2,\ldots,\tau_n) \prod_{r=1}^{n} u(t-\tau_r)\, d\tau_r \end{aligned} \qquad (1.1.1)$$

where $u(t)$ and $y(t)$ are the input and the output of the system, respectively, and $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ is known as the n^{th}-order Volterra kernel and belongs to a set of kernels which completely characterise the system output $y(t)$. The output $y(t)$ is given by the superposition of responses due to kernels of each order, as shown in Fig.1.1. It is well known that the first term in the above equation is the one-dimensional convolution integral used in linear system analysis and hence the Volterra functional series given by eqn.(1.1.1) is a generalisation of this linear convolution integral. The kernels are the properties of the system alone and are independent of the input signal.

1.2 Historical Review

The method of solution based on power series expansion[1], although covering a wider range of nonlinear problems, yields only the steady state response. However, in many practical problems, the transient response of the system is also desired and the need to obtain the steady state as well as transient response of a given system has led to an alternative method of analysis based on functional power series.

1.2.1 Functional Methods

The concept of functionals was first introduced by Volterra[2] early in the twentieth century and were used by him in the theory of elasticity. Wiener[3,4] used the theory of functionals in Brownian motion. Later, he applied the functional methods to study the response of a nonlinear electrical circuit[5] for a random input. He also used the functional representation to obtain a canonical form for nonlinear systems[4].

Bose[6] investigated the problem of canonical representation of nonlinear systems and suggested non-overlapping gate functions to obtain an orthogonal series representation of the system response for any input. The method of synthesising nonlinear Wiener filters using correlation functions and Fourier transforms, was extended by Chesler[7] to multi-input, multi-output systems, subjected to Gaussian input processes, using linear and zero-memory nonlinear subsystems. Schetzen[8] extended this work to systems with non-Gaussian inputs and obtained optimum filter-parameters by minimising the error between the filter output and the desired output. The nonlinear compensators for the synthesis of these systems were designed by Van Trees[9]. Recently, Reddy and Reddy[10] obtained a general expression for the functional expansion of an operator, which eliminates the use of special algorithms developed by Van Trees.

Barrett[11] used Volterra functional series to obtain the solution of a class of nonlinear equations by the method of successive approximations. The solution is convergent when the system nonlinearity is small. He also gave the expansions for cascade and inverse functionals.

Brilliant[12] introduced an important concept of an analytic system and gave a rigorous mathematical description of the theory of Volterra functionals. He also showed that the functional representation was well suited for the combination of nonlinear systems.

George[13] developed an operator algebra for various nonlinear system connections such as addition, multiplication, cascading and feedback combination, of two nonlinear systems. He introduced the multidimensional Laplace transforms and an "inspection technique" for the association-of-variables while inverting the multidimensional Laplace transforms.

The operator algebra was extended by Zames[14] for solving nonlinear feedback system equations by functional iteration. He introduced the theory of exponential iteration which may be used to study the realisability of nonlinear feedback systems.

McFee[15] obtained Volterra series solution using multidimensional Maclaurin series and used it to obtain the transient response of a nonlinear series-circuit containing a linear inductor and a nonlinear resistor.

Flake[16] developed a general method for obtaining the Volterra series solution of a large class of nonlinear systems with or without zero initial conditions. Bansal[17] extended this work to obtain explicit solutions of nonlinear multivariable differential and integro-differential equations, with non-zero initial conditions, using multidimensional Laplace transforms. Lubbock[18] presented a binomial theorem of operators, which simplifies the operator algebra used to obtain the Volterra functional expansion of various system combinations.

The recent contribution in this field has been made by Halme, Orava and Blomberg[19], who have introduced the concept of generalised polynomial operators and applied it to the theory of nonlinear systems.

1.2.2 Nonlinear Time Varying Systems

There has been considerable interest shown in the application of Volterra series for the analysis of nonlinear time-varying systems. Flake[20] extended his method of substitution to obtain the solution of nonlinear time varying differential equations, but this method loses practical importance because of difficulties involved in solving the resulting linear time varying partial differential equations. Ku and Su[21] have developed a method, using Barrett's method of successive approximations, for the Volterra series solution of a class of nonlinear time-varying equations. Since the analysis is carried out in time domain, the solution involves the evaluation of lengthy convolution integrals.

Ridings and Higgins[22] have presented an analysis method, using multidimensional Laplace transforms and the method of successive approximations, for obtaining the solution of a class of nonlinear time-varying systems. But, the solution converges slowly to the exact solution. Bansal[23] proposed a general method, using multilinear parametric transfer functions and Flake's method of substitution, for the analysis of a class of time-varying systems having polynomial-type nonlinearities and continuous time-varying parameters and showed that the Volterra series solution converges faster to the exact solution than that given by the method of successive approximations.

1.2.3 Stability and Convergence Properties of Volterra Series

A problem which is encountered in the Volterra series solution of nonlinear systems, when system nonlinearity is not small, is the convergence of the series. Alper[24] has demonstrated the difficulties that occur in the convergence of the Volterra series solution, by solving a second-order nonlinear differential equation. Barrett[25] has made an useful attempt to study the convergence properties of Volterra series solution of nonlinear problems. He determined the upper bound of each term of the Volterra series solution and obtained a majorant series by replacing each term of the Volterra series solution by its corresponding upper bound. The convergence of the Volterra series is then obtained from the convergence of the majorant series. The accuracy of this method depends upon how closely the majorant series approximates the original Volterra series.

Kuo and Wolf[26] have given convergence properties of Volterra series solutions for deterministic and random inputs. Parante[27] has given a sufficient condition for the convergence of Volterra series solution of a d.c shunt machine subjected to a bounded-input, in which the radius of convergence is obtained from a convergent geometric series, which dominates and identifies, term by term, the original Volterra series solution of the output(speed). Marchesini[28] has obtained a sufficient condition for the bounded-input bounded-output (BIBO) stability of a class of nonlinear unity feedback systems whose forward path consists of a zero-memory nonlinearity followed by a linear system. It states that the Volterra series solution of the response is bounded, for a bounded-input, if the supremum of the integral of the linear system weighting function is less than the reciprocal of the supremum of the distance function $\left|\dot{f}(e) - \dot{f}(0)\right|$, where $f(\cdot)$ represents the zero-memory nonlinearity and e is the error signal.

Christensen[29] has shown that a convergent Volterra series solution of a nonlinear system may be obtained using contraction mapping principle provided that the given system satisfies certain conditions. Trott and Christensen[30] have later shown that this Volterra series is unique. They also investigated, using Volterra series, the stability[31] of nonlinear systems which incorporate non-zero initial conditions. These methods, of finding the region of convergence, do not take into account the sign of the nonlinearity. To overcome this difficulty, Trott and Christensen[32], in another paper, have presented a method which expands the contraction mapping region, for a class of nonlinear systems, by providing a term for comparison with the slope of the nonlinearity and this provides a unique and convergent Volterra series solution of the given system.

Anjaiah and Reddy[33] have established an equivalence of multidimensional Laplace transforms and reversion methods of solution of nonlinear equations, which may enable one to predict the stability of the system.

1.2.4 Nonlinear System Identification

One of the properties of the Volterra functional series is that it lends itself for the analysis of systems with random inputs. One of the main problems in this application of the Volterra series is the explicit determination of the Volterra kernels of the system. Widnall[34] and Lee and Schetzen[35] have used Gaussian input signals and crosscorrelation measurements for the determination of kernels. The disadvantage of this method is that it requires an infinite crosscorrelation time. Huang[36] and Schetzen[37] have used multidimensional input impulses to measure the Volterra kernels of a given nonlinear system. Hooper and Gyftopoulos[38,39] have measured the first and second-order Volterra kernels using pseudo-random ternary signals, derived from m-sequence, and their correlation properties. They have also observed that, in the identification of nonlinear systems using pseudo-random signals, the autocorrelation functions of these signals contain undesirable non-zero values, but the most important and significant contribution towards this aspect was made by Barker and Pradisthayon[40], who also derived expressions for higher-order autocorrelation functions of pseudo-random signals of any number of levels. The recent contributions in this field were made by Barker and Obidegwu, who developed a combined cross-correlation method[41] for the measurement of second-order Volterra kernels. They also studied the effects of nonlinearities[42] on the estimates of system weighting functions obtained by crosscorrelation using pseudorandom signals. Sinha and Mahalanabis[43] developed, recently, recursive methods, using the techniques of adaptive Kalman filtering, for the identification of Volterra kernels of a nonlinear system.

The identification of nonlinear systems has also been carried out in frequency domain. Gardiner[44] proposed a method using sinusoidal input signals. Barker and Davy[45] have used pseudorandom signals and discrete Fourier transforms for the identification of linear systems and indicated that the presence of nonlinearities in the system introduce errors in the systems frequency response estimate.

1.2.5 Applications of Volterra Series to Practical Nonlinear Problems

The Volterra functional method has been extensively used in the past decade in solving multitude variety of nonlinear physical problems. Contos[46] used Volterra series in the simulation studies of respiratory mechanisms. Zames[47] used his operator theory for the analysis of nonlinear distortion in feedback amplifiers. Berger[48] has obtained the transient response of a closed loop system in which a nonlinear electrohydraulic control valve drives an inertia load through a hydraulic actuator. He has shown that Volterra series solution for a step input agrees with analogue computer results. Van Trees[49] has used Volterra functional series to study the nonlinear behaviour of phase locked loops for phase modulated signals.

Waddington and Fallside[50] have made a successful attempt, using Volterra functional series, to obtain a periodic solution of a second-order nonlinear relationship between the rotor angle of a synchronous machine and the input torque.

Lavi and Mastascusa[51] have analysed, using Volterra series, a class of nonlinear extremum seeking feedback systems in which the optimum performance is achieved by maximising the quadratic performance index. Roy and Sherman[52] have applied the theory of Volterra functionals to nonlinear system identification based on the techniques of pattern recognition and developed an on-line error correcting procedure to generate the desired Volterra kernels of a given system.

Narayanan[53,54,55] has analysed the problem of intermodulation distortion in transistor feedback and cascaded amplifiers, using Volterra series representation, and derived the closed-form expressions for overall gain and frequency dependent distortion to study the effects of various transistor parameters and the load and source parameters. In another paper, Maurer and Narayanan[56] have analysed the third-order model of the same transistor circuit with zero-mean Gaussian inputs.

An important and interesting contribution towards the application of the theory of Volterra functionals, was made by Stark[57], who characterised the human pupil, which is a complex neurological system, by first two Volterra kernels and measured the kernel functions using pseudorandom light excitation. This application shows, beyond any doubt, the practical utility of Volterra functional approach as a valuable method of analysis to various complex biological systems. Another interesting application was reported by Bansal[58], who used Volterra series to investigate the effect of temperature-feedback control on the growth of neutrons in a nuclear reactor and obtained a sufficient condition for the boundedness of the response of the reactor, for an arbitrary bounded input.

Neill[59] presented an improved method for analysing nonlinear electrical networks by formulating differential equations of circuit analysis as Volterra integral equations and used fast Fourier transform techniques to obtain their solution. He showed that the method is faster and more powerful than existing techniques, particularly when the steady-state response of an electrical network, to a periodic driving force, is required.

More recent contributions on the applications of Volterra functional series include that of Goldman[60] who provided a general description of inter-channel crosstalk created in a communication channel. This description is in the form of a Volterra series expansion of the interference signal in terms of the signal which produced the interference and from this description the intelligible and unintelligible parts of the crosstalk, for speech signals, are calculated. Narayanan and Poon[61] have analysed the distortion in bipolar transistors using a recently developed nonlinear device model, known as the integral charge control model, and Volterra series representation as a powerful analysis tool. Rice[62] has introduced double Volterra series to represent two-input, single-output nonlinear systems, with memory, and applied it to calculate the third-order distortion in a frequency converter. The problem of distortion in variable-capacitance diodes was investigated by Meyer and Stephens[63] using Volterra series. They obtained closed-form expressions for intermodulation distortion produced by variable capacitance diodes in series and parallel tuned circuits, and verified the results experimentally at frequencies upto 200MHz.

Volterra functional series has also been used in the most interesting field of binary data transmission. Lawless and Schwartz[64] have examined the transmission of binary data signals over channels containing quadratic nonlinearities and additive Gaussian noise, in which the samples of the received signal is represented by a discrete Volterra series. They derived optimum and sub-optimum receivers for such signals and examined their performance by means of computer simulations.

Landau and Leondes[65] have used Volterra series to represent radar angle-tracking loop, whose objective is to keep the target within the beamwidth of the radar antenna. The tracking performance is decided by the behaviour of the antenna pointing error. They used Volterra series approach to obtain approximate expressions for transient mean and variance of the antenna pointing error, as explicit functions of time, for a tracker with polynomial-type nonlinearities.

1.3 Discrete Nonlinear Systems

With the advent of high speed digital computers, the importance and significance of the study of discrete systems needs no special emphasis. A discrete system is a dynamic system, in which the variables take different values only at discrete instants of time, and may be a sampled-data system or a digital system. The former consists of a continuous dynamic system, characterised by a differential equation, with input normally applied through a sample and hold[66] device and the latter is a system, characterised by a difference equation, which operates on a discrete input, without a sample and hold device, to deliver a discrete output.

1.3.1 Historical Development

The classical methods of analysis of discrete nonlinear systems include describing function technique, phase plane method[66] and the discrete Liapunov functions[67]. Although the describing function method is useful for obtaining steady-state solutions and determining stability of systems in which higher harmonics are not pronounced, it becomes inconvenient when systems are cascaded. The phase plane method is convenient for obtaining transient solutions of second-order systems, even when the nonlinearities are not small, but becomes too cumbersome for higher-order systems. The discrete Liapunov functions are useful in investigating system stability, but it is not easy to determine a Liapunov function for a given system in a straightforward manner and the function is not useful in determining other system properties. However, the functional method is a very general method for representing nonlinear discrete systems. In the functional representation, a discrete system may be viewed as a "discrete functional" assigning a scalar value(output sequence) to each input sequence.

Bush[68] has presented some techniques for the synthesis of nonlinear systems. He also presented transform analysis, using multidimensional z transforms, for a class of nonlinear systems with sampled input and output. Alper[69] has used discrete Volterra series and multidimensional z transforms for the analysis of a class of nonlinear sampled-data systems, in which the nonlinearity in the form of instantaneous power series is followed by a sampler, this being the only class of systems in which the Volterra kernels take a form which is easily transformed. Lavi and Narayanan[70] have presented a method, using multidimensional modified z transforms, for obtaining the continuous output of a nonlinear sampled-data system. The systems they considered are the same as those considered by Alper. It should, however, be pointed out that the work reported by Alper, Bush and Lavi and Narayanan was done independently.

A notable contribution has come from Barker[71], who proposed a theorem for obtaining the synchronously sampled output of a nonlinear system subjected to a sampled-data input. Jagan and Reddy[72] have extended Bansal's[73] method for the solution of time-invariant nonlinear difference equations. Most recently, Fu and Farrison[74] have extended the work of Kuo and Wolf [26] to nonlinear discrete-time systems and developed recurrence relations for generating Volterra kernels in time and transform domains. They have also extended[75] the work of Marchesini and Picci[76] to nonlinear discrete systems. Padgett and Tsokos[77] have given, recently, the random solution of a stochastic discrete Volterra equation, using contraction mapping principle, and applied to nonlinear stochastic control systems.

In the field of analysis of nonlinear time-varying discrete systems, using discrete Volterra series, there have been only two contributions. One is by Bansal and Mali[78], who have extended Bansal's method[23] to analyse a class of time-varying systems using multidimensional parametric z transfer functions and applied to a practical time-varying discrete system involving a nonlinear element in its feedback path. The other paper is by Jagan and Desai[79], who have extended the methods of Bansal[23] and Ridings and Higgins[22], for the solution of nonlinear time varying discrete systems, and illustrated by means of an example. The example they considered is a feedback discrete system with a sinusoidal time-varying parameter and a second-order zero-memory nonlinearity in the feedback path.

The problem of convergence of discrete Volterra series pointed out by Alper[69] was investigated by Rao and Christensen[80], who have formulated a criterion for the convergence of the series using the contraction mapping principle. Rashed and Christensen[81] have extended this method, using Banach fixed-point theorem, to include non-zero initial conditions, for the Volterra series solution of a nonlinear difference equation representing a nonlinear discrete system.

1.3.2 Nonlinear Sampled-Data System

A typical nonlinear sampled-data system cascaded with a data-hold device is shown in Fig.1.2. This system is, currently, of practical importance as it allows a continuous nonlinear system to be simulated on a digital computer, which is often required for dynamic system investigations. If the continuous system with an input $z(t)$ is characterised by a set of Volterra kernels $w_n(\tau_1,\tau_2,\ldots,\tau_n)$, $n=1,2,\ldots,\infty$, as shown in Fig.1.2, then its output $y(t)$ is given by the Volterra series

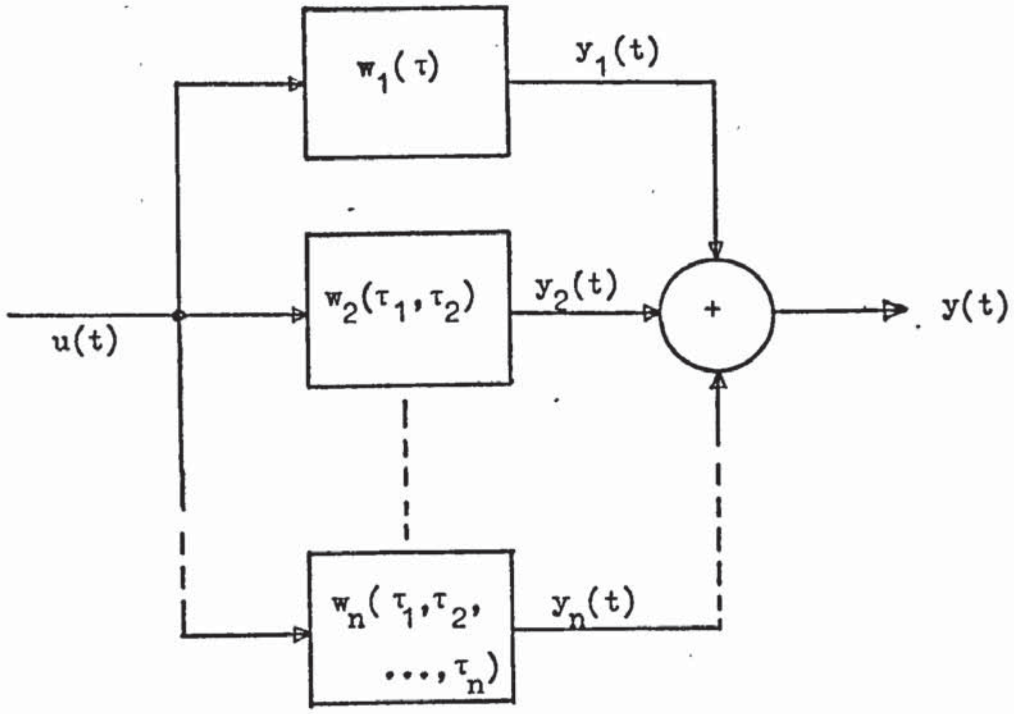


Fig.1.1 A nonlinear system characterised by a set of Volterra kernels.

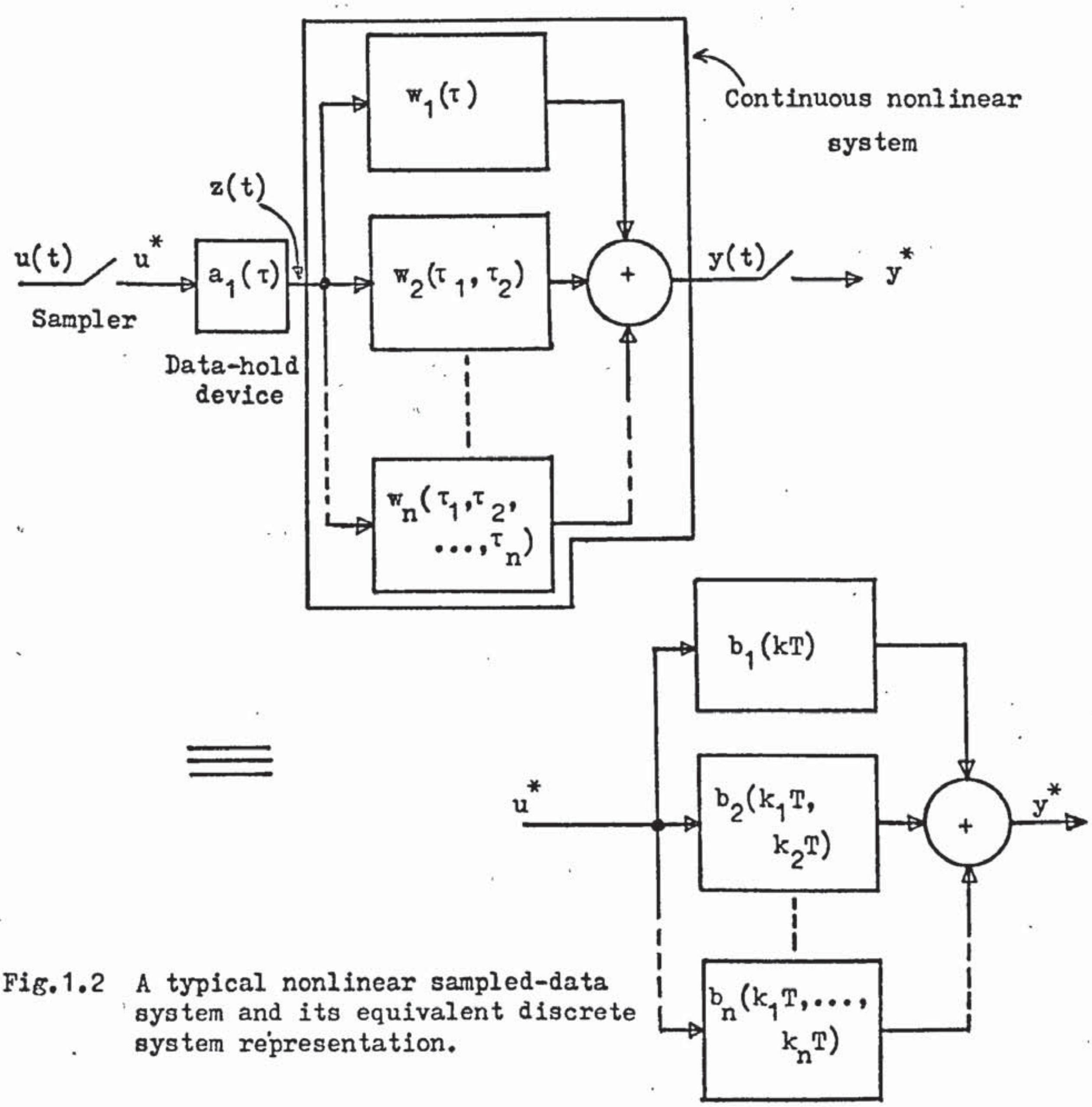


Fig.1.2 A typical nonlinear sampled-data system and its equivalent discrete system representation.

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w_n(\tau_1,\tau_2,\ldots,\tau_n) \prod_{r=1}^{n} z(t-\tau_r)\, d\tau_r \qquad (1.3.1)$$

For the sampled-data system, the sampled input $u^*(t)$, with sampling period T, is applied to the system through a linear data-hold device with impulse response $a_1(\tau)$, and the reconstructed signal z(t) is given by

$$z(t) = \int_{-\infty}^{\infty} a_1(\tau)\, u^*(t-\tau)\, d\tau \qquad (1.3.2)$$

where $u^*(t)$ is given by

$$u^*(t) = \sum_{K=-\infty}^{\infty} u(KT)\delta(t-KT) \qquad (1.3.3)$$

Then, from eqns.(1.3.1) to (1.3.3), the system output y(t) may be obtained as

$$y(t) = \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w_n(\tau_1,\tau_2,\ldots,\tau_n) \prod_{r=1}^{n} \int_{-\infty}^{\infty} a_1(\lambda_r)u^*(t-\tau_r-\lambda_r)d\tau_r\, d\lambda_r$$

$$= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} \left[\int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w_n(\lambda_1,\lambda_2,\ldots,\lambda_n) \prod_{r=1}^{n} a_1(\tau_r-\lambda_r)d\lambda_r\right] \times \prod_{r=1}^{n} u^*(t-\tau_r)d\tau_r$$

$$= \sum_{n=1}^{\infty} \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} b_n(\tau_1,\tau_2,\ldots,\tau_n) \prod_{r=1}^{n} \sum_{k_r=-\infty}^{\infty} u(k_rT)\delta(t-\tau_r-k_rT)\, d\tau_r$$

$$= \sum_{n=1}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} b_n(\langle t-k_1\rangle T, \langle t-k_2\rangle T, \ldots, \langle t-k_n\rangle T) \prod_{r=1}^{n} u(k_rT) \qquad (1.3.4)$$

$$\text{where } b_n(\tau_1,\tau_2,\ldots,\tau_n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w_n(\lambda_1,\lambda_2,\ldots,\lambda_n) \prod_{r=1}^{n} a_1(\tau_r-\lambda_r)\, d\lambda_r \qquad (1.3.5)$$

Thus, the synchronously sampled output values y(iT) of the nonlinear system may be obtained by letting t= iT, i= 0,1,2,...etc., in eqn.(1.3.4). Then y(iT) is given by the discrete Volterra series

$$y(iT) = \sum_{n=1}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} b_n(k_1T,k_2T,\ldots,k_nT) \prod_{r=1}^{n} u(\langle i-k_r\rangle T) \qquad (1.3.6)$$

where $b_n(k_1T,k_2T,\ldots,k_nT)$ is known as the n^{th}-order discrete Volterra kernel of the cascade combination of the data-hold device and the system.

For the particular case, when the sampled-data input u (kT) is applied to the nonlinear system through a zero-order hold[71], the synchronously sampled output of the system is given by discrete Volterra series

$$y(iT) = \sum_{n=1}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} p_n(k_1T,k_2T,\ldots,k_nT) \prod_{r=1}^{n} u(<i-k_r> T) \tag{1.3.7}$$

where $p_n(k_1T,k_2T,\ldots,k_nT)$ is the n^{th}-order discrete Volterra kernel of a nonlinear system cascaded with a zero-order hold and is given by

$$p_n(k_1T,k_2T,\ldots,k_nT) = \int_{(k_1-1)T}^{k_1T} \int_{(k_2-1)T}^{k_2T} \cdots \int_{(k_n-1)T}^{k_nT} w_n(\lambda_1,\lambda_2,\ldots,\lambda_n) \prod_{r=1}^{n} d\lambda_r , \tag{1.3.8}$$

since the output z(t) of the zero-order hold is given by

$$z(t) = \int_0^T u(t-\tau)\, d\tau \tag{1.3.9}$$

Hence, the synchronous sampling theorem[71] states that, for a nonlinear system with a sampled-data input applied through a data-hold device, the synchronously sampled output is given by the discrete Volterra series whose coefficients are the discrete Volterra kernels of the cascade combination of the data-hold device and the system. The synchronous sampling theorem may be extended to obtain the asynchronous sampled output of a nonlinear system cascaded with a data-hold device, by letting $t=(i+m)T$, $0 \leq m < 1$, in eqn.(1.3.4).

The discrete Volterra kernels of different order give an explicit input-output relation for a discrete system which is very useful for block diagram manipulations. Another advantage of the discrete Volterra series representation is that it allows the use of random inputs, which is highly desirable for the dynamic analysis of systems. Hence, the Volterra functional approach is a useful and powerful tool for the analysis and synthesis of nonlinear continuous and discrete systems.

1.4 Transform Methods for Nonlinear System Analysis

It is well known that, for a time invariant linear system, the output signal y(t) for a given input signal u(t), can be obtained by means of a convolution integral given by

$$y(t) = \int_{-\infty}^{\infty} w_1(\tau)\, u(t-\tau)\, d\tau \qquad (1.4.1)$$

where $w_1(\tau)$ is the system weighting function.

In practice, this method of obtaining the system output is used much less frequently than the corresponding method involving Laplace transforms because the 'convolution in time domain becomes a simple multiplication in the frequency domain'. Hence, eqn.(1.4.1), in Laplace transforms, is given by

$$Y(s) = W_1(s)\, U_1(s) \qquad (1.4.2)$$

and $y(t)$ can be easily obtained from (1.4.2) using inverse Laplace transform, where $W_1(s)$ is the system transfer function and $Y(s)$ and $U_1(s)$ are Laplace transforms of $y(t)$ and $u(t)$, respectively.

In practice, this procedure is preferred for the analysis of linear systems, because, (a)comprehensive tables of transform pairs are available, (b)the procedures are essentially algebraic and (c)the system transfer function $W_1(s)$ may be synthesised directly from the basic physical laws of the processes involved and the system structure.

The development of transform methods for nonlinear systems depends on these advantages for its own applications.

For a continuous nonlinear system with a given input $u(t)$, the output $y(t)$ is given by eqn.(1.1.1), provided that the set of kernels $w_n(\tau_1, \tau_2, \ldots, \tau_n)$, $n= 1,2,\ldots$, are known and in order to apply transform methods, it is necessary to use multidimensional Laplace transforms. George[13] made the first attempt to define multidimensional Laplace transforms and used them to transform the higher order terms of eqn.(1.1.1). He also suggested an "inspection rule" for associating the complex variables while inverting the multidimensional Laplace transforms.

For nonlinear discrete systems, the discrete output $y(kT)$ for a discrete input $u(iT)$, is given by

$$y(iT) = \sum_{k=-\infty}^{\infty} w_1(kT)\, u(\langle i-k\rangle T) + \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} w_2(k_1T, k_2T) \prod_{r=1}^{2} u(\langle i-k_r\rangle T) + \ldots$$

$$= \sum_{n=1}^{\infty} \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} w_n(k_1T, k_2T, \ldots, k_nT) \prod_{r=1}^{n} u(\langle i-k_r \rangle T) \tag{1.4.3}$$

where $w_n(k_1T, k_2T, \ldots, k_nT)$ is an n^{th} order discrete Volterra kernel and in order to apply transform methods, it is necessary to use multidimensional z transforms. The multidimensional z transforms play the same role as the multidimensional Laplace transforms in continuous nonlinear systems. Alper[82] suggested the use of multidimensional z transforms for the analysis of nonlinear discrete systems and it may be used to obtain the transform of higher order terms in eqn.(1.4.3). Thus, for nonlinear discrete systems, the theory involving multidimensional pulse transfer functions may be easily developed in a way analogous to the continuous case.

The more difficult, but common, problem is that of a continuous nonlinear system subjected to a sampled-data input, and the output synchronously sampled. For a nonlinear sampled-data system shown in Fig.1.2, the synchronously sampled output $y(kT)$ for a given sampled-data input $u(iT)$, may be obtained from eqn.(1.3.4), provided that the set of discrete kernels, $b_n(k_1T, k_2T, \ldots, k_nT)$, $1 \leqslant n < \infty$, are known. The method suggested by Alper may be used to obtain the multidimensional z transform of eqn.(1.3.4). But, this procedure leads to a cumbersome process and it is shown later that the multidimensional z transform of a kernel may be obtained from the multidimensional Laplace transform of the kernel, which is easily synthesised for a large class of nonlinear systems[88]. The time-domain and transform-domain relationships for continuous and sampled-data nonlinear systems, are shown in Fig.1.3.

1.5 Outstanding Problems

The uptodate review of the literature on the theory and application of Volterra functionals, reveals that the application of multidimensional transform methods for the analysis of continuous nonlinear systems has yielded useful results. In contrast, the application of multidimensional transform methods to sampled-data nonlinear systems has, in practice, been restricted to relatively simple class of systems[68,69,70] in which the non-

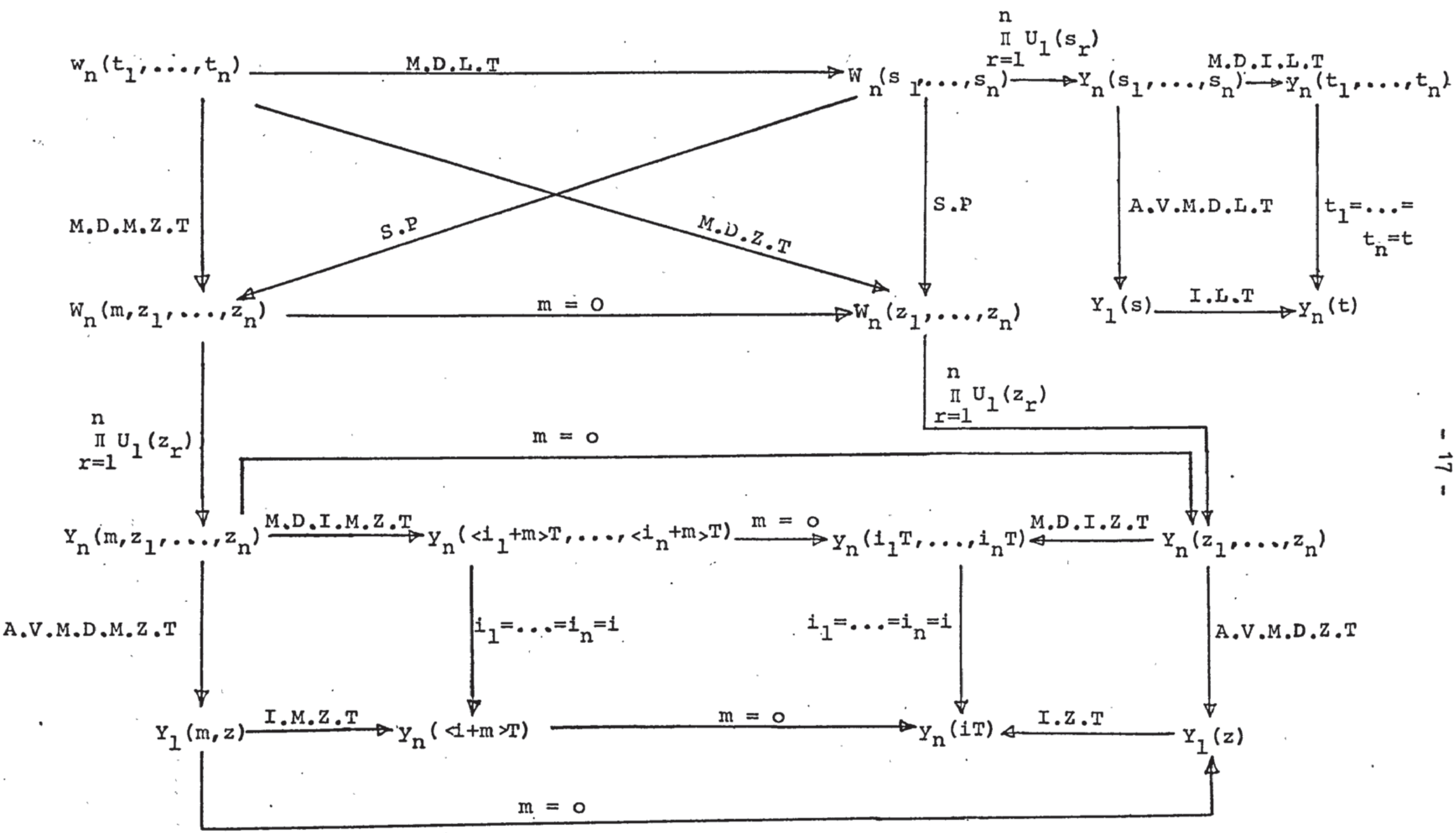


Fig.1.3 Picture showing the time domain and transform domain relationship for nonlinear continuous and sampled-data systems.

linearity in the form of an instantaneous power series is followed by a sampler, this being the only class of system in which the Volterra kernels take a form which is easily transformed. This is because of the difficulty in obtaining the multidimensional z transform of a nonlinear system with kernels not separated by a sampler.

A problem which occurs frequently in dynamic system investigations is that of simulating a continuous system by means of a discrete system. This often requires a data-hold device to be associated with the continuous system, so that a discrete simulator for the cascade combination of the data-hold device and the continuous system may be obtained. For nonlinear continuous systems cascaded with a data-hold device, a method to obtain the multidimensional z transform of the cascade combination is, therefore, required. A synthesis procedure to obtain the discrete simulator from this multidimensional z transform is also required.

The use of the transform methods in the investigation of the stability of nonlinear sampled-data systems characterised by discrete Volterra series is not completely explored.

The advantages of the state variable method over the conventional transfer function approach in linear system theory are well known. Many practical systems have more than one input and one output. A need, therefore, exists for a unified representation of nonlinear continuous and digital systems, single-variable and multivariable systems, in state space. The development of a method for the solution of the state variable problems of these systems, using multidimensional transform methods, is also a useful contribution.

1.6 The Scope of the Present Investigation

The aim of the present investigation is to solve some of the problems mentioned in the previous section and to make the theory of Volterra functionals, using multidimensional transform methods, more useful and easily applicable to a large class of continuous and discrete nonlinear systems.

A sequential process based on simple procedures is described for obtaining the multidimensional z transform of a nonlinear-system kernel from the multidimensional Laplace transform of the kernel, which is easily synthesised for a large class of nonlinear systems. For nonlinear systems cascaded with a data-hold device, a method is developed by which the multidimensional z transform of the cascade combination may be obtained by applying the process to a derived multidimensional Laplace transform. A sequential process based on association-of-variables procedure is also described for obtaining the sampled-data output of a nonlinear system with a given sampled-data input. The analysis of nonlinear systems with asynchronous input-output sampling is investigated using multidimensional modified z transforms. The theory of operator algebra is extended to nonlinear discrete systems which may include zero-order subsystems. A set of six basic operations is described by which the generalised Volterra functional expansion of various nonlinear sampled-data system connections, in which the inputs to subsystems are applied through data-hold devices, may be obtained.

A systematic procedure is developed for the synthesis of second and third-order discrete Volterra kernels using a finite number of linear discrete systems and multipliers. This procedure is used to obtain a discrete simulator for a continuous nonlinear system preceded by a data-hold device and it is indicated how the number of multipliers may be reduced.

The formulation and solution of state variable problem of single-variable and multivariable systems with multiplicative, functional or polynomial type nonlinearities, is investigated using multidimensional Laplace transforms. The solution of the state variable problem is then extended to nonlinear sampled-data systems cascaded with data-hold devices.

The nonlinear models of a practical feedback FM demodulator are derived using Volterra series and transform methods, for the measurement of distortion and crosstalk in the demodulator. The spectral results obtained here by digital simulation of the model, are compared with those obtained by others in order to establish the superiority of this method in its generality and in achieving better results.

CHAPTER 2

NONLINEAR SAMPLED-DATA SYSTEM ANALYSIS BY MULTIDIMENSIONAL Z TRANSFORMS

2.1 Introduction

It is shown in the previous chapter that the input-output relation of the sampled-data nonlinear system is given by the discrete form of Volterra series, in which the kernels are the discrete equivalent of the Volterra kernels of the continuous system. However, this characterisation of nonlinear sampled-data systems by discrete Volterra series may be conveniently represented in the transform domain using multidimensional z transform. The multidimensional z transforms characterise nonlinear sampled-data systems in the way one-dimensional z transforms characterise the linear sampled-data systems.

This chapter develops a method for the analysis of nonlinear sampled-data systems, by multidimensional z transforms, in which the continuous system is represented by a Volterra functional series. A sequential process based on simple procedures is described for obtaining the multidimensional z transform(M.D.Z.T) of a discrete Volterra kernel from the multidimensional Laplace transform(M.D.L.T) of the continuous system kernel. The method is illustrated by deriving the two dimensional z transform, of a nonseparable 2^{nd}-order kernel.

For nonlinear systems with sampled input applied through a data-hold device, a method is developed by which the M.D.Z.T of the kernel of the cascade combination of the nonlinear system and the data-hold device may be obtained by applying the sequential process to a derived M.D.L.T. To illustrate the method, the two-dimensional z transform of a 2^{nd}-order nonlinear system preceded by a zero-order hold is derived.

A sequential process based on association-of-variables procedure is also developed for obtaining the sampled-data output of a nonlinear system with a given sampled-data input and is illustrated by obtaining the associated output z transform of a 3^{rd}-order nonlinear system.

2.2 Nonlinear System Equations and Transforms

In this section, a brief exposition of the theory of Volterra series is given for mainly two types of nonlinear systems: (a) Continuous and (b) Sampled-data systems.

2.2.1 Continuous Systems

The explicit input-output relationship for a time-invariant continuous nonlinear system may be expressed in the form of Volterra functional series[13] as

$$y(t) = \sum_{n=1}^{\infty} \underbrace{\int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty}}_{n} w_n(\tau_1, \tau_2, \ldots \tau_n) \prod_{r=1}^{n} u(t-\tau_r)d\tau_r \tag{2.2.1}$$

where $w_n(\tau_1, \tau_2, \ldots, \tau_n)$ is the n^{th} order Volterra kernel of the system. Since the output of the system is given by the superposition of responses due to each kernel, it is enough if the response $y_n(t)$ due to the n^{th} order kernel is only considered. The response due to n^{th} order kernel is then given by

$$y_n(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w_n(\tau_1, \tau_2, \ldots, \tau_n) \prod_{r=1}^{n} u(t-\tau_r)d\tau_r \quad .$$

or

$$= \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} w_n(t-\tau_1, t-\tau_2, \ldots, t-\tau_n) \prod_{r=1}^{n} u(\tau_r)d\tau_r \tag{2.2.2}$$

The M.D.L.T of a continuous-time function $f_n(t_1, t_2, \ldots, t_n)$ is defined by[13]

$$F_n(s_1, s_2, \ldots, s_n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty} \cdots \int_{-\infty}^{\infty} f_n(t_1, t_2, \ldots, t_n) \prod_{r=1}^{n} e^{-s_r t_r} dt_r \tag{2.2.3}$$

and hence, to transform eqn.(2.2.2), it is necessary to define a function $y_n(t_1, t_2, \ldots, t_n)$, by introducing a set of artificial variables $t_1, t_2, \ldots, t_n$ into $y_n(t)$, such that

$$y_n(t) = \Big[y_n(t_1, t_2, \ldots, t_n)\Big]_{t_1=t_2=\ldots=t_n=t} \tag{2.2.4}$$

Then,

$$y_n(t_1,t_2,\ldots,t_n) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w_n(t_1-\tau_1,t_2-\tau_2,\ldots,t_n-\tau_n)\prod_{r=1}^{n} u(\tau_r)d\tau_r \tag{2.2.5}$$

and its M.D.L.T is given by

$$Y_n(s_1,s_2,\ldots,s_n) = W_n(s_1,s_2,\ldots,s_n)\prod_{r=1}^{n} U_1(s_r) \tag{2.2.6}$$

The advantages of using eqn.(2.2.6) in preference to eqn.(2.2.2) for system analysis are that, for a large class of nonlinear systems, $W_n(s_1,s_2,\ldots,s_n)$ may be easily synthesised by simple algebraic procedures, rather than obtained from the cumbersome forms of $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ and also $y_n(t)$ may be obtained from $Y_n(s_1,s_2,\ldots,s_n)$ fairly easily by association-of-variables procedures[73].

2.2.2 Sampled-data Systems

For a sampled-data input $u^*(t)$ given by

$$u^*(t) = u(t)\sum_{k=-\infty}^{\infty}\delta(t-kT)$$

or

$$= \sum_{k=-\infty}^{\infty} u(kT)\,\delta(t-kT) \tag{2.2.7}$$

where T is the sampling period, the continuous response of the n^{th} order kernel is obtained from eqns.(2.2.2) and (2,2.7) as

$$y_n(t) = \int_{-\infty}^{\infty}\int_{-\infty}^{\infty}\cdots\int_{-\infty}^{\infty} w_n(t-\tau_1,t-\tau_2,\ldots,t-\tau_n)\prod_{r=1}^{n}\sum_{k_r=-\infty}^{\infty} u(k_rT)\,\delta(\tau_r-k_rT)d\tau_r$$

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} w_n(t-k_1T,t-k_2T,\ldots,t-k_nT)\prod_{r=1}^{n} u(k_rT) \tag{2.2.8}$$

The synchronously sampled output of the subsystem is, therefore, given by

$$y_n(iT) = \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} w_n(\langle i-k_1\rangle T,\langle i-k_2\rangle T,\ldots,\langle i-k_n\rangle T)\prod_{r=1}^{n} u(k_rT)$$

or

$$= \sum_{k_1=-\infty}^{\infty}\sum_{k_2=-\infty}^{\infty}\cdots\sum_{k_n=-\infty}^{\infty} w_n(k_1T,k_2T,\ldots,k_nT)\prod_{r=1}^{n} u(\langle i-k_r\rangle T) \tag{2.2.9}$$

where $w_n(k_1T, k_2T, \ldots, k_nT)$ is the n^{th} order discrete Volterra kernel and the analogy with eqn.(2.2.2) is obvious.

The M.D.Z.T of a function $f_n(t_1, t_2, \ldots, t_n)$ is defined by

$$F_n(z_1, z_2, \ldots, z_n) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_n=-\infty}^{\infty} f_n(i_1T, i_2T, \ldots, i_nT) \prod_{r=1}^{n} z_r^{-i_r} \tag{2.2.10}$$

and, to transform eqn.(2.2.9), it is therefore necessary to introduce a set of artificial variables $i_1, i_2, \ldots, i_n$ into $y_n(iT)$ such that

$$y_n(iT) = \left[y_n(i_1T, i_2T, \ldots, i_nT)\right]_{i_1=i_2= \cdots =i_n=i} \tag{2.2.11}$$

Further, since $w_n(i_1T, i_2T, \ldots, i_nT)$ is a realisable kernel, which is zero for all $i_r<0$, $r = 1,2,\ldots,n$, eqn.(2.2.9) becomes

$$y_n(i_1T, i_2T, \ldots, i_nT) = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} w_n(<i_1-k_1>T, <i_2-k_2>T, \ldots, <i_n-k_n>T) \prod_{r=1}^{n} u(k_rT) \tag{2.2.12}$$

Then, the M.D.Z.T of eqn.(2.2.12) is given by

$$Y_n(z_1, z_2, \ldots, z_n) = W_n(z_1, z_2, \ldots, z_n) \prod_{r=1}^{n} U_1(z_r) \tag{2.2.13}$$

It is advantageous to use eqn.(2.2.13) in preference to eqn.(2.2.9) only if $W_n(z_1, z_2, \ldots, z_n)$ may be easily derived and if $y_n(iT)$ may be obtained from $Y_n(z_1, z_2, \ldots, z_n)$ by simple association-of-variables procedures.

2.3 Multidimensional Z Transforms of Volterra Kernels

For most of the systems, the procedures for first obtaining $w_n(\tau_1, \tau_2, \ldots, \tau_n)$ and then determining $W_n(z_1, z_2, \ldots, z_n)$ through eqn.(2.2.10) are complex and impractical. On the other hand, since $W_n(s_1, s_2, \ldots, s_n)$ may be more easily obtained than $w_n(\tau_1, \tau_2, \ldots, \tau_n)$ for a large class of nonlinear systems, this function may be taken as the starting point for the derivation of $W_n(z_1, z_2, \ldots, z_n)$. The basis for such a derivation lies in the sequential nature of multidimensional transforms. The function $W_n(s_1,s_2,\ldots,s_n)$ may be obtained from $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ by a sequential process of n stages, each of which involves the Laplace transformation of a function of a single τ variable, the remaining τ and s variables being taken as constant. Similarly, the function $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ may be obtained from $W_n(s_1,s_2,\ldots,s_n)$ by a sequential process of n stages, each of which involves the inverse Laplace transformation of a function of a single s variable, the remaining s and τ variables being taken as constant. By similar procedures, $W_n(z_1,z_2,\ldots,z_n)$ can be obtained from $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ using n z transformations and $w_n(i_1T,i_2T,\ldots,i_nT)$ can be obtained from $W_n(z_1,z_2,\ldots,z_n)$ using n inverse z transformations. The variables in these sequential process may be taken in any order. Since the function $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ is a realisable kernel, which is zero for all $\tau_i<0$, $i=1,2,\ldots,n$, only unilateral Laplace and z transforms are used throughout in this chapter. The sequential process for obtaining $W_n(z_1,z_2,\ldots,z_n)$ from $W_n(s_1,s_2,\ldots,s_n)$ is now described and is illustrated by obtaining $W_2(z_1,z_2)$ of a nonlinear system with nonseparable 2^{nd} order kernel.

2.3.1 The Sequential Process

The first stage in the process for obtaining $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ from $W_n(s_1,s_2,\ldots,s_n)$ may be written as

$$W_{n-1}(\tau_1,s_2,s_3,\ldots,s_n) = \frac{1}{2\pi j}\int_{c_1-j\infty}^{c_1+j\infty} W_n(s_1,s_2,\ldots,s_n)\, e^{s_1\tau_1}\, ds_1$$

$$s_2,s_3,\ldots,s_n \text{ constant} \qquad (2.3.1)$$

where c_1 is a real number in the half-plane of convergence of $W_n(s_1,s_2,\ldots,s_n)$. Instead of continuing this process, the first stage in the process for obtaining $W_n(z_1,z_2,\ldots,z_n)$ from $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ is now applied to the function $W_{n-1}(\tau_1,s_2,s_3,\ldots,s_n)$. This gives

$$
\begin{aligned}
{}^{L}_{Z}W_{\substack{n-1\\1}}(z_1,s_2,s_3,\ldots,s_n) &= \sum_{i_1=0}^{\infty} W_{n-1}(i_1T,s_2,s_3,\ldots,s_n)\, z_1^{-i_1} \\
&= \frac{1}{2\pi j}\int_{c_1-j\infty}^{c_1+j\infty} W_n(s_1,s_2,\ldots,s_n) \sum_{i_1=0}^{\infty} e^{s_1 i_1 T}\, z_1^{-i_1}\, ds_1 \\
&= \frac{1}{2\pi j}\int_{c_1-j\infty}^{c_1-j\infty} \frac{W_n(s_1,s_2,\ldots,s_n)}{(1-e^{s_1T}z_1^{-1})}\, ds_1
\end{aligned}
$$

$$
s_2,s_3,\ldots,s_n \text{ constant} \qquad (2.3.2)
$$

which defines a mixed transform ${}^{L}_{Z}W_{\substack{n-1\\1}}(z_1,s_2,s_3,\ldots,s_n)$. On repeating this procedure for the variable s_2 in ${}^{L}_{Z}W_{\substack{n-1\\1}}(z_1,s_2,s_3,\ldots,s_n)$, a further mixed transform ${}^{L}_{Z}W_{\substack{n-2\\2}}(z_1,z_2,s_3,\ldots,s_n)$ is obtained, and so on, until $W_n(z_1,z_2,z_3,\ldots,z_n)$ is finally obtained as required.

This procedure, then, defines the required sequential process of n stages for obtaining $W_n(z_1,z_2,\ldots,z_n)$ from $W_n(s_1,s_2,\ldots,s_n)$, for which, the procedure at each stage is given by

$$
\begin{aligned}
&{}^{L}_{Z}W_{\substack{n-r\\r}}(z_1,z_2,\ldots,z_r,s_{r+1},s_{r+2},\ldots,s_n) \\
&\quad = \frac{1}{2\pi j}\int_{c_r-j\infty}^{c_r+j\infty} \frac{{}^{L}_{Z}W_{\substack{n-r+1\\r-1}}(z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n)}{(1-e^{s_rT}z_r^{-1})}\, ds_r
\end{aligned}
$$

$$
\begin{gathered}
z_1,z_2,\ldots,z_{r-1},s_{r+1},s_{r+2},\ldots,s_n \text{ constant,} \\
r = 1,2,\ldots,n
\end{gathered} \qquad (2.3.3)
$$

where c_r is a real number in the half-plane of convergence of ${}^{L}_{Z}W_{\substack{n-r+1\\r-1}}(z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n)$. If ${}^{L}_{Z}W_{\substack{n-r+1\\r-1}}(z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n)$ has no branch points, then the above integral may be evaluated using Cauchy's residue theorem as

$$
\begin{aligned}
&{}^{L}_{Z}W_{\substack{n-r\\r}}(z_1,z_2,\ldots,z_{r-1},z_r,s_{r+1},s_{r+2},\ldots,s_n) \\
&\quad = \sum_{\text{residues}} \frac{{}^{L}_{Z}W_{\substack{n-r+1\\r-1}}(z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n)}{(1-e^{s_rT}z_r^{-1})}
\end{aligned}
$$

$$z_1, z_2, \ldots, z_{r-1}, s_{r+1}, s_{r+2}, \ldots, s_n \text{ constant} \qquad r = 1,2,\ldots,n \tag{2.3.4}$$

This procedure is the very simple one of determining the z transform, with argument z_r, from the corresponding Laplace transform with argument s_r, the remaining s and z variables being taken as constant. In many cases of practical interest, this process is reduced to an inspection procedure using tables of related Laplace and z transforms. In the sequential process defined here, the variables are taken in a certain order for convenience, but, in general, the variables can be taken in any order. The formula for the complete process may be obtained, from eqn.(2.3.3), as

$$W_n(z_1, z_2, \ldots, z_n) = \frac{1}{(2\pi j)^n} \int_{c_1-j\infty}^{c_1+j\infty} \int_{c_2-j\infty}^{c_2+j\infty} \cdots \int_{c_n-j\infty}^{c_n+j\infty} W_n(s_1, s_2, \ldots, s_n) \prod_{r=1}^{n} \frac{ds_r}{(1 - e^{s_r T} z_r^{-1})} \tag{2.3.5}$$

and this integral may be evaluated by the repeated use of the residue theorem for which, the procedure at each stage is given by eqn.(2.3.4).

For a class of nonlinear systems shown in Fig.2.1(a), where the kernels are separable, an explicit functional relationship exists between $W_n(s_1, s_2, \ldots, s_n)$ and $W_n(z_1, z_2, \ldots, z_n)$. If $w_n(\tau_1, \tau_2, \ldots, \tau_n)$ is given by

$$w_n(\tau_1, \tau_2, \ldots, \tau_n) = \prod_{r=1}^{n} w_1(\tau_r)$$

then, its transforms are also separable and are given by

$$W_n(s_1, s_2, \ldots, s_n) = \prod_{r=1}^{n} W_1(s_r) \quad , \; W_n(z_1, z_2, \ldots, z_n) = \prod_{r=1}^{n} W_1(z_r) \tag{2.3.6}$$

$$\text{where } W_1(s_r) = \int_{-\infty}^{\infty} w_1(t_r)\, e^{-s_r t_r}\, dt_r, \text{ and } W_1(z_r) = \frac{1}{2\pi j} \int_{c_r-j\infty}^{c_r+j\infty} \frac{W_1(s_r)}{(1 - e^{s_r T} z_r^{-1})}\, ds_r \tag{2.3.7}$$

However, if the kernels are separated by a sampler as shown in Fig.2.1(b), then the functional relationship between the kernel of the cascade and its z transform are given by

$$w_n(l_1 T, l_2 T, \ldots, l_n T) = \sum_{i=0}^{\min(l_r)} k_1(iT) \prod_{r=1}^{n} j_1(\langle l_r - i \rangle T)$$

$$W_n(z_1,z_2,\ldots,z_n) = K_1(\prod_{r=1}^{n} z_r) \prod_{r=1}^{n} J_1(z_r) \qquad (2.3.8)$$

For other systems, i.e., for nonlinear systems with kernels not separated by a sampler as shown in Fig.2.2, the sequential process must be used. For this system, the overall kernel and its transform are given by

$$w_n(\tau_1,\tau_2,\ldots,\tau_n) = \int_0^{\min(\tau_r)} k_1(\lambda) \prod_{r=1}^{n} j_1(\tau_r-\lambda)\, d\lambda$$

$$W_n(s_1,s_2,\ldots,s_n) = K_1(s_1+s_2+\ldots+s_n) \prod_{r=1}^{n} J_1(s_r) \qquad (2.3.9)$$

Then, using the sequential process, $W_n(z_1,z_2,\ldots,z_n)$ may be obtained.

2.3.2 Example - Nonlinear System with Nonseparable 2^{nd} Order Kernel

To illustrate the use of the sequential process, consider the nonlinear system with nonseparable 2^{nd} order kernel shown in Fig.2.3. For this system, $W_2(s_1,s_2)$ is easily obtained[13] as

$$W_2(s_1,s_2) = \frac{\alpha^2\beta}{(s_1+s_2+b)(s_1+a)(s_2+a)}$$

$$= \frac{\alpha^2\beta}{(s_2+a)(s_2+b-a)}\left[\frac{1}{(s_1+a)} - \frac{1}{(s_1+s_2+b)}\right] \qquad (2.3.10)$$

Applying the sequential process to $W_2(s_1,s_2)$, ${}_{\substack{L\\Z}}W_{\substack{1\\1}}(z_1,s_2)$ is obtained as

$${}_{\substack{L\\Z}}W_{\substack{1\\1}}(z_1,s_2) = \sum_{\text{residues}} \frac{W_2(s_1,s_2)}{(1-e^{s_1T}z_1^{-1})}, \quad s_2 \text{constant}$$

$$= \frac{\alpha^2\beta}{(s_2+a)(s_2+b-a)}\left[\frac{1}{(1-e^{-aT}z_1^{-1})} - \frac{1}{(1-e^{-(s_2+b)T}z_1^{-1})}\right]$$

$$= \frac{\alpha^2\beta e^{-aT}z_1^{-1}(1-e^{-(b-a)T}e^{-s_2T})}{(b-2a)(1-e^{-aT}z_1^{-1})(1-e^{-bT}z_1^{-1}e^{-s_2T})}\left[\frac{1}{(s_2+a)} - \frac{1}{(s_2+b-a)}\right]$$

Then, $W_2(z_1,z_2)$ may be obtained by applying the sequential process to ${}_{\substack{L\\Z}}W_{\substack{1\\1}}(z_1,s_2)$, as

$$W_2(z_1,z_2) = \sum_{\text{residues}} \frac{{}_{\substack{L\\Z}}W_{\substack{1\\1}}(z_1,s_2)}{(1-e^{s_2T}z_2^{-1})}, \quad z_1 \text{constant}$$

$$= \frac{e^{-aT}z_1^{-1}(1-e^{-(b-a)T}z_2^{-1})\alpha^2\beta}{(b-2a)(1-e^{-aT}z_1^{-1})(1-e^{-bT}z_1^{-1}z_2^{-1})}\left[\frac{1}{(1-e^{-aT}z_2^{-1})} - \frac{1}{(1-e^{-(b-a)T}z_2^{-1})}\right]$$

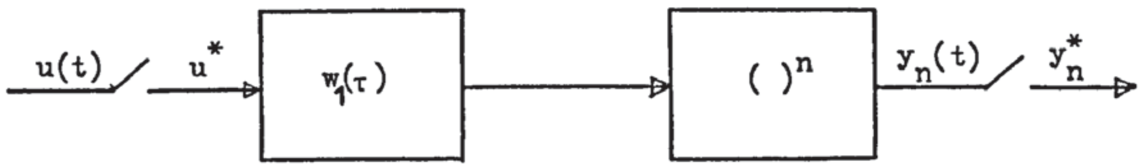


Fig.2.1(a) Nonlinear system with separable n^{th}-order kernel.

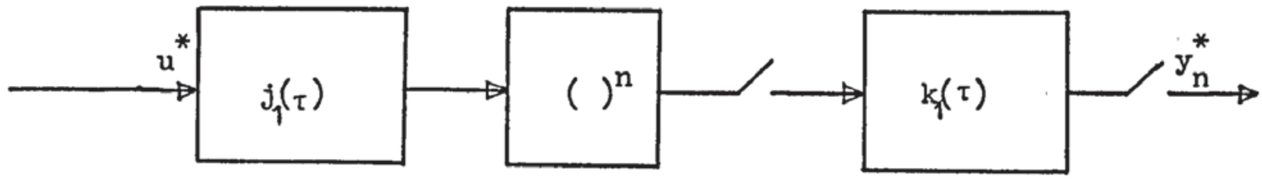


Fig.2.1(b) Nonlinear system with kernels separated by a sampler.

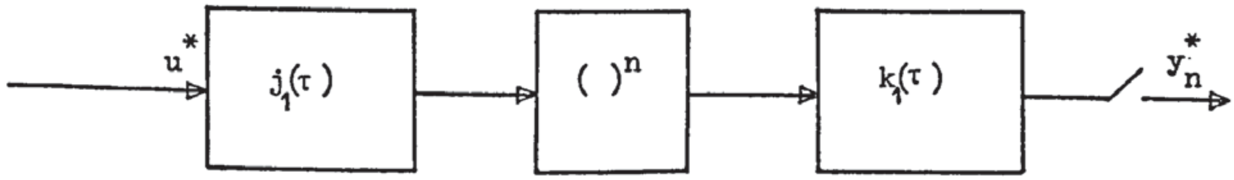


Fig.2.2 Nonlinear system with kernels not separated by a sampler.

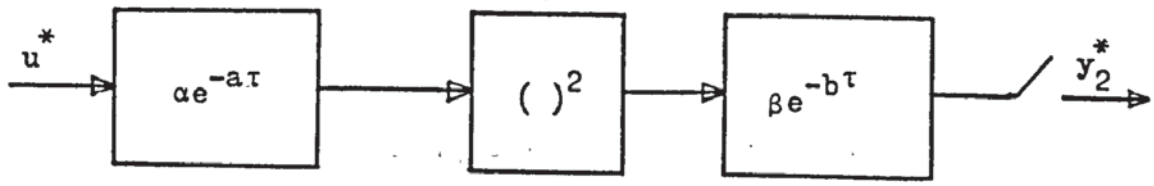


Fig.2.3 Nonlinear system with nonseparable 2^{nd}-order kernel.

$$= \frac{(e^{-2aT} - e^{-bT})z_1 z_2 \alpha^2 \beta}{(b-2a)(z_1 z_2 - e^{-bT})(z_1 - e^{-aT})(z_2 - e^{-aT})} , \qquad (2.3.11)$$

which is the required result.

2.4 Analysis of Systems Preceded by Data-Hold Devices

The analysis of sampled-data systems is of little practical significance unless the data-hold devices are considered. An n^{th}-order Volterra kernel $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ cascaded with a data-hold device with impulse response $a_1(\tau)$, is shown in Fig.1.2. If $A_1(s)$ is the transfer function of the data-hold device, then the M.D.L.T of the cascade combination of the data-hold device and the kernel is given by

$$B_n(s_1,s_2,\ldots,s_n) = W_n(s_1,s_2,\ldots,s_n) \prod_{r=1}^{n} A_1(s_r) \qquad (2.4.1)$$

and the M.D.Z.T of the output sequence $y^*(t)$ is given by

$$Y_n(z_1,z_2,\ldots,z_n) = B_n(z_1,z_2,\ldots,z_n) \prod_{r=1}^{n} U_1(z_r) \qquad (2.4.2)$$

where $B_n(z_1,z_2,\ldots,z_n)$ is the M.D.Z.T of the cascade combination of the n^{th} order kernel and the data-hold device and may be obtained by applying the sequential process to $B_n(s_1,s_2,\ldots,s_n)$. In this section, it is shown how the two frequently used data-hold devices, namely, zero-order hold and first-order hold, may be introduced into the analysis of nonlinear sampled-data systems.

2.4.1 Zero-Order Hold

Consider a nonlinear system with an n^{th} order kernel $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ preceded by a zero-order hold. The transform of the zero-order hold is given by

$$A_1(s) = \frac{(1- e^{-sT})}{s}$$

and hence the M.D.L.T of the cascade combination of the kernel and the zero-order hold is given by

$$P_n(s_1,s_2,\ldots,s_n) = W_n(s_1,s_2,\ldots,s_n) \prod_{r=1}^{n} \frac{(1-e^{-s_rT})}{s_r} \tag{2.4.3}$$

Then, the multidimensional z transform $P_n(z_1,z_2,\ldots,z_n)$ of the cascade combination of the kernel and the zero-order hold, may be obtained from $P_n(s_1,s_2,\ldots,s_n)$, using the sequential process.

An alternative treatment of a nonlinear system with kernel $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ preceded by a zero-order hold is shown in Fig.2.4. The transform of the zero-order hold is treated as a mixed transform $(\frac{z-1}{z}).\frac{1}{s}$, and, since $W_n(s_1,s_2,\ldots,s_n)$ is preceded by $1/s$, the M.D.L.T of their cascade combination is given by

$$H_n(s_1,s_2,\ldots,s_n) = W_n(s_1,s_2,\ldots,s_n) \prod_{r=1}^{n} (\frac{1}{s_r}) \tag{2.4.4}$$

The corresponding $H_n(z_1,z_2,\ldots,z_n)$ is then obtained by applying the sequential process to $H_n(s_1,s_2,\ldots,s_n)$. Since $H_n(z_1,z_2,\ldots,z_n)$ is preceded by $(\frac{z-1}{z})$, the M.D.Z.T of their cascade combination gives the M.D.Z.T of the nonlinear system with n^{th} order kernel preceded by a zero-order hold, as

$$P_n(z_1,z_2,\ldots,z_n) = H_n(z_1,z_2,\ldots,z_n) \prod_{r=1}^{n} \frac{(z_r-1)}{z_r} \tag{2.4.5}$$

2.4.2 Example - Nonlinear System with 2^{nd} Order Kernel preceded by Zero-order Hold

The method developed here is illustrated by considering a 2^{nd}-order nonlinear system preceded by a zero-order hold, as shown in Fig.2.5. For this system, $H_2(s_1,s_2)$ is given by

$$H_2(s_1,s_2) = \frac{W_2(s_1,s_2)}{s_1s_2}$$

$$= \frac{\alpha^2\beta}{(s_1+s_2+b)(s_1+a)(s_2+a)s_1s_2} \tag{2.4.6}$$

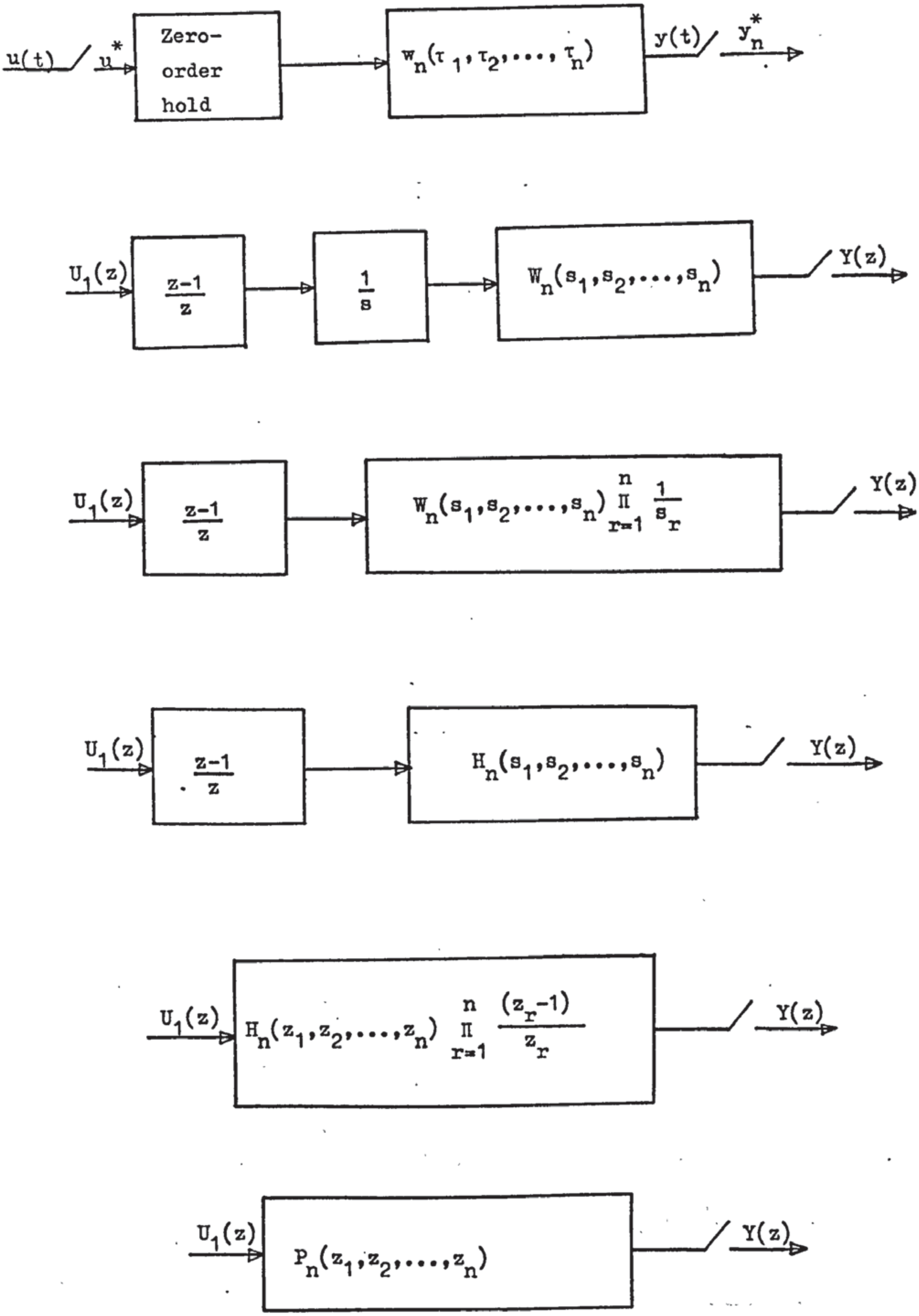


Fig.2.4 General treatment of nonlinear system with n^{th}-order kernel preceded by a zero-order hold.

where $W_2(s_1,s_2)$ of eqn.(2.3.10) has been used. Then, ${}_{\substack{L\\Z}}H_{\substack{1\\1}}(z_1,s_2)$ is obtained, using the sequential process, as

$$ {}_{\substack{L\\Z}}H_{\substack{1\\1}}(z_1,s_2) = \sum_{\text{residues}} \frac{H_2(s_1,s_2)}{(1 - e^{s_1T} z_1)} \quad , \; s_2 \text{ constant} $$

$$ = \alpha^2\beta z_1 \left[\frac{(1 - e^{-aT})}{a(z_1-1)(z_1- e^{-aT})s_2(s_2+a)(s_2+b-a)} - \frac{(1 - e^{-bT}e^{-s_2T})}{(z_1-1)(z_1- e^{-bT}e^{-s_2T})s_2(s_2+a)(s_2+b)(s_2+b-a)} \right] $$

Applying the sequential process to ${}_{\substack{L\\Z}}H_{\substack{1\\1}}(z_1,s_2)$ gives $H_2(z_1,z_2)$ as

$$ H_2(z_1,z_2) = \sum_{\text{residues}} \frac{{}_{\substack{L\\Z}}H_{\substack{1\\1}}(z_1,s_2)}{(1 - e^{s_2T} z_2^{-1})} \; , \; z_1 \text{ constant} $$

$$ = \frac{\alpha^2\beta \begin{bmatrix} 2a\{a(1-e^{-bT}) - b(1-e^{-aT})\}(z_1-e^{-aT})(z_2-e^{-aT}) \\ +ab(1-e^{-aT})(e^{-bT}-e^{-aT})\{(z_1-e^{-aT})+(z_2-e^{-aT})\} \\ +b(b-a)(1-e^{-aT})^2(z_1z_2- e^{-bT}) \end{bmatrix} z_1z_2}{a^2b(b-a)(b-2a)(z_1z_2-e^{-bT})(z_1-e^{-aT})(z_2-e^{-aT})(z_1-1)(z_2-1)} $$

and therefore the M.D.Z.T of the cascade combination is given by

$$ P_2(z_1,z_2) = H_2(z_1,z_2) \prod_{r=1}^{2} \frac{(z_r-1)}{z_r} $$

$$ = \frac{\begin{bmatrix} 2a\{a(1-e^{-bT}) - b(1-e^{-aT})\}(z_1-e^{-aT})(z_2-e^{-aT}) \\ +ab(1-e^{-aT})(e^{-bT}-e^{-aT})\{(z_1-e^{-aT})+(z_2-e^{-aT})\} \\ +b(b-a)(1-e^{-aT})^2(z_1z_2-e^{-bT}) \end{bmatrix} \alpha^2\beta}{a^2b(b-a)(b-2a)(z_1z_2-e^{-bT})(z_1-e^{-aT})(z_2-e^{-aT})} , \qquad (2.4.7) $$

which is the required result.

2.4.3 First-order Hold

The treatment of a nonlinear system with kernel $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ preceded by a first-order hold is shown in Fig.2.6. The mixed transform of the first-order hold[83] is given by $(\frac{z-1}{z})^2.\frac{(1+sT)}{s^2T}$, the M.D.L.T of their cascade combination is given by

$$G_n(s_1,s_2,\ldots,s_n) = W_n(s_1,s_2,\ldots,s_n) \prod_{r=1}^{n} \frac{(1+s_rT)}{s_r^2T} \tag{2.4.8}$$

The corresponding $G_n(z_1,z_2,\ldots,z_n)$ is then obtained from $G_n(s_1,s_2,\ldots,s_n)$ by the sequential process developed in section 2.3. Since $G_n(z_1,z_2,\ldots,z_n)$ is preceded by $(\frac{z-1}{z})^2$, the M.D.Z.T of their cascade combination is the M.D.Z.T of a nonlinear system preceded by a first-order hold and is given by

$$Q_n(z_1,z_2,\ldots,z_n) = G_n(z_1,z_2,\ldots,z_n) \prod_{r=1}^{n} (\frac{z_r-1}{z_r})^2 \tag{2.4.9}$$

2.5 Association of Variables in Multidimensional Z Transforms

In order to obtain the sampled-data output $y^*(t)$ of a nonlinear system in response to a sampled-data input $u^*(t)$, it is necessary to use the multidimensional inverse z transformation to obtain

$$y_n(i_1T,i_2T,\ldots,i_nT) = (\frac{1}{2\pi j})^n \oint_{C_1} \oint_{C_2} \cdots \oint_{C_n} Y_n(z_1,z_2,\ldots,z_n) \prod_{r=1}^{n} z_r^{i_r-1} dz_r$$

$$= \frac{1}{(2\pi j)^n} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_n} W_n(z_1,z_2,\ldots,z_n) \times \prod_{r=1}^{n} U_1(z_r)\, z_r^{i_r-1} dz_r \tag{2.5.1}$$

If the system is assumed to be stable in the bounded-input bounded-output (BIBO) sense, then the contour C_r for $1 \leq r \leq n$, will be the unit circle in the z_r plane, indented to include singularities on the unit circle. Singularities of $Y_n(z_1,z_2,\ldots,z_n)$ are within C_r if the input $u(iT)$ and its z transform $U_1(z)$ are not bilateral. However, if $u(iT)$

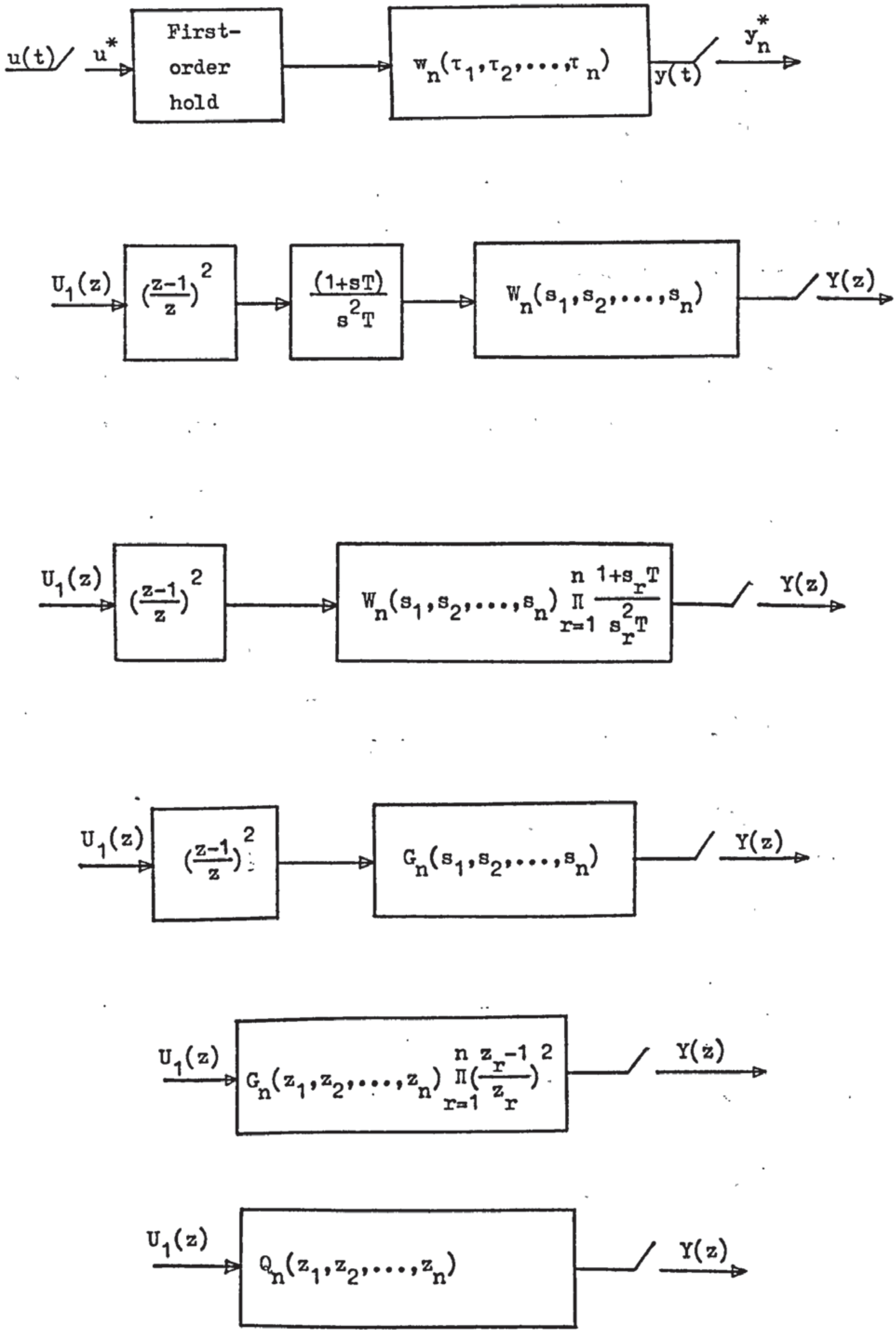


Fig.2.6 General treatment of n^{th}-order nonlinear system preceded by a first-order hold.

and $U_1(z)$ are bilateral, then the singularities of $Y_n(z_1,z_2,\dots,z_n)$ without C_r may be introduced through $U_1(z_r)$. Thus, the response at the sampling instants $y_n(iT)$ may be obtained from eqns.(2.5.1) and (2.2.11). To use eqn.(2.5.1) directly and then obtain $y_n(iT)$ using eqn.(2.2.11) is an extremely cumbersome process involving the derivation of mixed transforms in z and i variables. A more practical approach is to develop an association-of-variables procedure, for associating the complex variables $z_1,z_2,\dots,z_n$ in $Y_n(z_1,z_2,\dots,z_n)$, analogous to the procedure[73,84] developed for multidimensional Laplace transforms.

2.5.1 A Sequential Process for Association Of Variables

Consider first the association of the variables z_n and z_{n-1}. Letting $i_n = i_{n-1}$ in eqn.(2.5.1) gives

$$y_n(i_1T,i_2T,\dots,i_{n-1}T,i_{n-1}T)$$

$$= \frac{1}{(2\pi j)^{n-1}} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_{n-2}} \left[\oint_{C_{n-1}} \left\{ \frac{1}{2\pi j} \oint_{C_n} Y_n(z_1,z_2,\dots,z_{n-1},z_n)\,dz_n \right\} \times (z_{n-1}z_n)^{i_{n-1}-1}\, dz_{n-1} \right] \prod_{r=1}^{n-2} z_r^{i_r-1}\, dz_r \tag{2.5.2}$$

Setting $z_{n-1} = \frac{z_{n-1}}{z_n}$ and $dz_{n-1} = \frac{dz_{n-1}}{z_n}$ in the above equation yields

$$y_n(i_1T,i_2T,\dots,i_{n-1}T,i_{n-1}T)$$

$$= \frac{1}{(2\pi j)^{n-1}} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_{n-1}} \left\{ \frac{1}{2\pi j} \oint_{C_n} Y_n\left(z_1,z_2,\dots,\frac{z_{n-1}}{z_n},z_n\right) \frac{1}{z_n}\, dz_n \right\} \times \prod_{r=1}^{n-1} z_r^{i_r-1}\, dz_r$$

$$= \frac{1}{(2\pi j)^{n-1}} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_{n-1}} Y_{n-1}(z_1,z_2,\dots,z_{n-1}) \prod_{r=1}^{n-1} z_r^{i_r-1}\, dz_r$$

$$= y_n(i_1T,i_2T,\dots,i_{n-1}T) \tag{2.5.3}$$

This defines a procedure for associating z_n and z_{n-1} by replacing

z_{n-1} by $(\frac{z_{n-1}}{z_n})$ in the n dimensional transform $Y_n(z_1,z_2,\ldots,z_n)$ to obtain the (n-1) dimensional associated transform $Y_{n-1}(z_1,z_2,\ldots,z_{n-1})$ as

$$Y_{n-1}(z_1,z_2,\ldots,z_{n-1}) = \frac{1}{2\pi j}\oint_{C_n} Y_n(z_1,z_2,\ldots,z_{n-2},\frac{z_{n-1}}{z_n},z_n)\,\frac{dz_n}{z_n} \qquad (2.5.4)$$

By associating the variables z_{n-1} and z_{n-2} in $Y_{n-1}(z_1,z_2,\ldots,z_{n-1})$ using a similar procedure, the (n-2) dimensional associated transform $Y_{n-2}(z_1,z_2,\ldots,z_{n-2})$ of $y_n(i_1T,i_2T,\ldots,i_{n-2}T)$ is then obtained and so on until the unidimensional associated z transform $Y_1(z_1)$ of $y_n(iT)$ is obtained, and dropping the suffix 1 gives the z transform $Y_1(z)$ of $y_n(iT)$ as required. This defines a sequential process of (n-1) stages, for which the procedure at each stage is given by

$$Y_{r-1}(z_1,z_2,\ldots,z_{r-1})=\frac{1}{2\pi j}\oint_{C_r} Y_r(z_1,z_2,\ldots,z_{r-2},\frac{z_{r-1}}{z_r},z_r)\,\frac{1}{z_r}\,dz_r \qquad (2.5.5)$$

$$z_1,z_2,\ldots,z_{r-1} \text{ constant,}$$
$$r= n,n-1,\ldots,2$$

If $Y_r(z_1,z_2,\ldots,z_{r-2},\frac{z_{r-1}}{z_r},z_r)$ has no branch points, this equation may be written as

$$Y_{r-1}(z_1,z_2,\ldots,z_{r-1}) = \sum_{\substack{\text{residues}\\ \text{within } C_r}} Y_r(z_1,z_2,\ldots,z_{r-2},\frac{z_{r-1}}{z_r},z_r)\,\frac{1}{z_r} \qquad (2.5.6)$$

$$z_1,z_2,\ldots,z_{r-1} \text{ constant}$$
$$r= n,n-1,\ldots,2$$

which can be evaluated using the residue theorem.

It should be noted that if $Y_r(z_1,z_2,\ldots,z_r)$ is an unilateral z transform, then it has no singularities without C_{r-1} and therefore $Y_r(z_1,z_2,\ldots,z_{r-2},\frac{z_{r-1}}{z_r},z_r)$ has no singularities within C_r arising from the term $(\frac{z_{r-1}}{z_r})$. On the other hand, if $Y_r(z_1,z_2,\ldots,z_r)$ is a bilateral z transform, then it may have singularities without C_{r-1}, in which case $Y_r(z_1,z_2,\ldots,z_{r-2},\frac{z_{r-1}}{z_r},z_r)$ will have singularities within C_r arising from the term $\frac{z_{r-1}}{z_r}$ and these singularities will also contribute residues in eqn.(2.5.6).

From eqn.(2.5.5), a formula defining the complete sequential process is

$$Y_1(z_1) = \frac{1}{(2\pi j)^{n-1}} \oint_{C_2} \oint_{C_3} \ldots \oint_{C_n} Y_n\left(\frac{z_1}{z_2}, \frac{z_2}{z_3}, \frac{z_3}{z_4}, \ldots, \frac{z_{n-1}}{z_n}, z_n\right) \times \prod_{r=2}^{n} \frac{1}{z_r}\, dz_r \qquad (2.5.7)$$

This integral can be evaluated by using eqns.(2.5.5) and (2.5.6) successively. In this sequential process, the z variables have been associated in a certain order for convenience, but in general, the variables may be associated in any order. If the variables z_n and z_1 are associated first, then the variables z_{n-1} and z_1 and so on, an alternative sequential process is defined:

$$Y_1(z_1) = \frac{1}{(2\pi j)^{n-1}} \oint_{C_2} \oint_{C_3} \ldots \oint_{C_n} Y_n\left(\frac{z_1}{\prod_{p=2}^{n} z_p}, z_2, z_3, \ldots, z_n\right) \prod_{r=2}^{n} \frac{1}{z_r}\, dz_r \qquad (2.5.8)$$

for which the procedure at each stage is given by

$$Y_{r-1}(z_1,z_2,\ldots,z_{r-1})$$

$$= \frac{1}{2\pi j} \oint_{C_r} Y_r\left(\frac{z_1}{\prod_{p=2}^{r} z_p}, z_2,z_3,\ldots,z_r\right) \frac{1}{z_r}\, dz_r$$

$$= \sum_{\substack{\text{residues} \\ \text{within } C_r}} Y_r\left(\frac{z_1}{\prod_{p=2}^{r} z_p}, z_2,z_3,\ldots,z_r\right) \frac{1}{z_r}$$

$$z_1,z_2,\ldots,z_{r-1} \text{ constant}$$
$$r = n,n-1,\ldots,2 \tag{2.5.9}$$

and this can be evaluated by residues as before. In general, with this sequential process, the variables can be associated with any selected variable in any order.

Special Cases

(a) Inspection Technique

If, at any stage in the sequential process, $Y_r(z_1,z_2,\ldots,z_r)$ may be expressed in the form as

$$Y_r(z_1,z_2,\ldots,z_r) = \frac{\Psi(z_1,z_2,\ldots,z_{r-2},z_{r-1}z_r)}{(z_{r-1}-e^{-aT})(z_r-e^{-bT})} \tag{2.5.10}$$

then an inspection technique, similar to that used for multidimensional Laplace transforms[13], may be developed to associate the variables z_r and z_{r-1}. Using eqn.(2.5.6), the associated transform can be obtained as

$$Y_{r-1}(z_1,z_2,\ldots,z_{r-1}) = \sum_{\substack{\text{residues}\\ \text{within } C_r}} \frac{\Psi(z_1,z_2,\ldots,z_{r-2},z_{r-1})}{\left(\frac{z_{r-1}}{z_r} - e^{-aT}\right)(z_r - e^{-bT})\, z_r}$$

$$= \Psi(z_1,z_2,\ldots,z_{r-1}) \left[\frac{1}{(z_{r-1} - e^{-aT}z_r)(z_r - e^{-bT})}\right]_{\substack{\text{residue}\\ \text{at } z_r=e^{-bT}}}$$

$$= \frac{\Psi(z_1,z_2,\ldots,z_{r-1})}{(z_{r-1} - e^{-(a+b)T})}, \tag{2.5.11}$$

(b) Separable Case

If, at any stage in the sequential process, $Y_r(z_1,z_2,\ldots,z_r)$ may be expressed in the separable form as

$$Y_r(z_1,z_2,\ldots,z_r) = \prod_{p=1}^{r} Y_1(z_p) \tag{2.5.12}$$

then the inverse transform, $y_r(i_1T,i_2T,\ldots,i_rT)$ is also separable and is given by

$$y_r(i_1T, i_2T, \ldots, i_rT) = \prod_{p=1}^{r} y_1(i_pT) \tag{2.5.13}$$

where $y_1(i_pT)$ is the inverse z transform of $Y_1(z_p)$. Equating $i_1 = i_2 = \ldots = i_r = i$ in eqn.(2.5.13) results in

$$y_r(iT) = \left[y_1(iT)\right]^r \tag{2.5.14}$$

and therefore, the associated z transform becomes

$$Y_1(z) = \sum_{i=0}^{\infty} y_r(iT)\, z^{-i} , \tag{2.5.15}$$

Thus, the problem of finding the associated transform for the separable case is reduced to a sequence of one dimensional problems of finding z and inverse z transforms.

In particular, if

$$Y_r(z_1, z_2, \ldots, z_r) = \prod_{p=1}^{r} \frac{z_p}{(z_p - e^{-a_pT})} , \tag{2.5.16}$$

then the associated transform is given by

$$Y_1(z) = \frac{z}{(z - \prod_{p=1}^{r} e^{-a_pT})} \tag{2.5.17}$$

After obtaining the associated transform $Y_1(z)$ of $Y_n(z_1, z_2, \ldots, z_n)$, the systems response at the sampling instants may then be obtained, from the one dimensional inverse transform of $Y_1(z)$ as

$$y_n(iT) = \frac{1}{2\pi j} \oint_C Y_1(z)\, z^{i-1}\, dz \tag{2.5.18}$$

2.5.2 Example - Nonlinear System with 3^{rd} Order Kernel

To illustrate the sequential process developed for the association-of-variables, consider the system shown in Fig.2.7, for which the M.D.L.T is

$$W_3(s_1,s_2,s_3) = \frac{\alpha^3\beta}{(s_1+s_2+s_3+b)(s_1+a)(s_2+a)(s_3+a)}$$

The M.D.Z.T of the system can be obtained by applying the sequential process of section 2.3, to $W_3(s_1,s_2,s_3)$. The result is

$$W_3(z_1,z_2,z_3) = \frac{\alpha^3\beta(e^{-3aT} - e^{-bT})\ z_1z_2z_3}{(b-3a)(z_1- e^{-aT})(z_2- e^{-aT})(z_3- e^{-aT})(z_1z_2z_3- e^{-bT})}$$

If the input to this system is a unit impulse with transform 1, then, from eqn.(2.2.13), the output is given by

$$Y_3(z_1,z_2,z_3) = \frac{\alpha^3\beta\,(e^{-3aT}- e^{-bT})\ z_1z_2z_3}{(b-3a)(z_1- e^{-aT})(z_2- e^{-aT})(z_3- e^{-aT})(z_1z_2z_3- e^{-bT})} \tag{2.5.19}$$

Associating z_3 and z_2 by inspection technique gives

$$Y_2(z_1,z_2) = \frac{\alpha^3\beta\,(e^{-3aT}- e^{-bT})\ z_1z_2}{(b-3a)(z_1z_2- e^{-bT})(z_1- e^{-aT})(z_2- e^{-2aT})}$$

and associating z_2 and z_1 by inspection technique gives

$$Y_1(z_1) = \frac{\alpha^3\beta\,(e^{-3aT}- e^{-bT})\ z_1}{(b-3a)(z_1- e^{-bT})(z_1- e^{-3aT})} \tag{2.5.20}$$

which is the required output transform.

If, however, the input is a sampled step with z transform $U_1(z) = \frac{z}{z-1}$, then the output is given by

$$Y_3(z_1,z_2,z_3) = \frac{\alpha^3\beta\ (e^{-3aT}- e^{-bT})\ z_1^2z_2^2z_3^2}{(b-3a)(z_1-1)(z_1- e^{-aT})(z_2-1)(z_2- e^{-aT})(z_3-1)(z_3- e^{-aT}) \times (z_1z_2z_3-e^{-bT})} \tag{2.5.21}$$

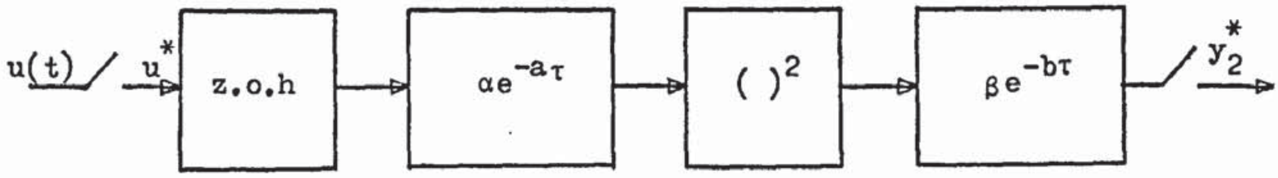


Fig.2.5 Nonlinear system with 2nd-order kernel preceded by a zero-order hold.

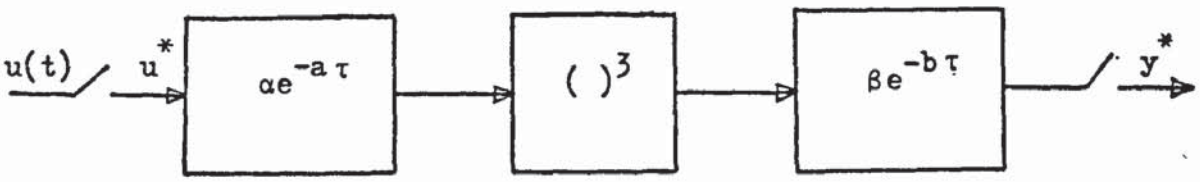


Fig.2.7 Nonlinear system with 3rd-order kernel.

and the inspection technique can not be used. Associating the variables z_3 and z_2 using eqn.(2.5.6) gives

$$Y_2(z_1,z_2) = \frac{\alpha^3\beta\,(e^{-3aT}-e^{-bT})\;z_1^2z_2^2}{(b-3a)(z_1z_2-e^{-bT})(z_1-1)(z_1-e^{-aT})}$$

$$\times \sum_{\substack{\text{residues}\\ \text{within } C_3}} \frac{1}{(\frac{z_2}{z_3}-1)(\frac{z_2}{z_3}-e^{-aT})(z_3-1)(z_3-e^{-aT})z_3}$$

$$= \frac{\alpha^3\beta\,(e^{-3aT}-e^{-bT})\;z_1^2z_2^2}{(b-3a)(z_1z_2-e^{-bT})(z_1-1)(z_1-e^{-aT})}$$

$$\times \left[\frac{z_3}{(z_2-z_3)(z_2-z_3e^{-aT})(z_3-1)(z_3-e^{-aT})}\right]_{\substack{\text{residues}\\ z_3=1\\ z_3=e^{-aT}}}$$

$$= \frac{\alpha^3\beta\,(e^{-3aT}-e^{-bT})(z_2+e^{-aT})\;z_1^2z_2^2}{(b-3a)(z_1z_2-e^{-bT})(z_1-1)(z_1-e^{-aT})(z_2-1)(z_2-e^{-aT})(z_2-e^{-2aT})}$$

Associating the variables z_2 and z_1 similarly gives

$$Y_1(z) = \frac{\alpha^3\beta(e^{-3aT}-e^{-bT})\{z^2+2ze^{-aT}(1+e^{-aT})+e^{-3aT}\}z^2}{(b-3a)(z-e^{-bT})(z-1)(z-e^{-aT})(z-e^{-2aT})(z-e^{-3aT})} \qquad (2.5.22)$$

which is the required output transform.

2.6 Conclusions

The multidimensional z transform of a nonlinear system kernel can be obtained by applying the sequential process developed here, to the multidimensional Laplace transform of the kernel, which is easily synthesised for a large class of nonlinear systems characterised by Volterra series. The procedure at each stage of this process is a simple one, which, in many cases, may be carried out by inspection or the calculation of residues. The multidimensional z transform of a nonlinear system cascaded with a data-hold device can be obtained by this sequential process applied to the derived multidimensional Laplace transform of the cascade.

The sampled-data output of a nonlinear system with a given sampled-data input may be obtained by the sequential process based on the association-of-variables procedure, developed here. This procedure at each stage is also a simple one, which may be carried out by inspection or the calculation of residues, in many cases. For an n-dimensional z transform, the process, in general, is to be applied (n-1) times successively to obtain a unidimensional associated z transform $Y_1(z)$.

CHAPTER 3

ANALYSIS OF ASYNCHRONOUS SAMPLED-DATA NONLINEAR SYSTEMS BY MULTIDIMENSIONAL MODIFIED Z TRANSFORMS

3.1 Introduction

Even though multidimensional z transforms are useful for obtaining the output at sampling instants, it provides no information about the output between the sampling instants. In sampled-data systems, the need often arises for finding the output between the sampling instants. For the analysis of such systems, multidimensional modified z transforms may be used similar to the modified z transforms used in linear system analysis. In this chapter, an attempt is made to develop the theory of multidimensional modified z transform(M.D.M.Z.T) for the analysis of sampled-data systems, in which the instants of sampling the output are not synchronous with the occurrence of the input impulses but have the same period.

Some new theorems are developed for the association-of-variables in M.D.M.Z.T by which the continuous output of the system may be obtained. Finally, the use of the transform methods is illustrated by means of two examples. Some useful properties and theorems of M.D.M.Z.T are described in Appendix A.3.1 and the corresponding properties of the M.D.Z.T may then be obtained by letting m=0. The tables of multidimensional z and modified z transforms and their associated transforms are given in Appendix A.3.2.

3.2 Mutidimensional Modified Z Transforms

Multidimensional modified z transforms characterise asynchronous sampled-data nonlinear systems in the same way as the modified z transforms characterise asynchronous linear sampled-data systems. In this section, the method of obtaining the M.D.M.Z.T of the Volterra kernel of a continuous system is described and is extended for the case when data-hold devices are cascaded with the system.

3.2.1 Volterra Series Representation of Asynchronous Sampled-data Systems

An asynchronous sampled-data system is a sampled-data system with a fictitious predictor($e^{mT}, 0 \leq m < 1$) introduced at its output, as shown in Fig.3.1. The output of the predictor is sampled to yield $y(\langle i+m \rangle T)$ which provides all the information about the output between the sampling instants if the dummy variable m is varied between 0 and 1.

The output $y_n(t)$ of the n^{th} order kernel $w_n(\tau_1, \tau_2, \ldots, \tau_n)$ for a sampled-data input $u^*(t)$ is given by eqn.(2.2.8). The sampled output of the fictitious predictor then becomes

$$y_n(\langle i+m \rangle T) = \sum_{k_1=-\infty}^{\infty} \sum_{k_2=-\infty}^{\infty} \cdots \sum_{k_n=-\infty}^{\infty} w_n(\langle i+m-k_1 \rangle T, \langle i+m-k_2 \rangle T, \ldots, \langle i+m-k_n \rangle T) \times \prod_{r=1}^{n} u(k_r T)$$

$$0 \leq m < 1 \qquad (3.2.1)$$

The multidimensional modified z transform $F_n(m, z_1, z_2, \ldots, z_n)$ of a function $f_n(t_1, t_2, \ldots, t_n)$ is defined here as

$$F_n(m, z_1, z_2, \ldots, z_n) = \sum_{i_1=-\infty}^{\infty} \sum_{i_2=-\infty}^{\infty} \cdots \sum_{i_n=-\infty}^{\infty} f_n(\langle i_1+m \rangle T, \langle i_2+m \rangle T, \ldots, \langle i_n+m \rangle T) \prod_{r=1}^{n} z_r^{-i_r}$$

$$0 \leq m < 1 \qquad (3.2.2)$$

where it should be noted that the use of dummy variables $m_1, m_2, \ldots, m_n$

in a multidimensional modified z transform, as suggested by Bush[69] and applied by others to the case of nonlinear systems with separable kernels[70], is not in fact necessary. If $f_n(i_1T, i_2T, \ldots, i_nT) = 0$ for all $i_r < 0$, $r = 1,2,\ldots,n$, then the range of summation in eqn.(3.2.2) is from 0 to ∞ and $F_n(m, z_1, z_2, \ldots, z_n)$ is called the multidimensional unilateral modified z transform of $f_n(t_1, t_2, \ldots, t_n)$. It should be noted that by letting m=0, eqn.(3.2.2) reduces to

$$F_n(0, z_1, z_2, \ldots, z_n) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f_n(i_1T, i_2T, \ldots, i_nT) \prod_{r=1}^{n} z_r^{-i_r}$$

$$= F_n(z_1, z_2, \ldots, z_n) \tag{3.2.3}$$

which is the M.D.Z.T of the function $f_n(t_1, t_2, \ldots, t_n)$, and with m=1, eqn.(3.2.2) yields

$$F_n(1, z_1, z_2, \ldots, z_n) = \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} f_n(<i_1+1>T, <i_2+1>T, \ldots, <i_n+1>T) \times \prod_{r=1}^{n} z_r^{-i_r}$$

$$= \left(\prod_{r=1}^{n} z_r\right) \left[F_n(z_1, z_2, \ldots, z_n) - f_n(0)\right] \tag{3.2.4}$$

where, in arriving at eqn.(3.2.4), forward shifting theorem has been used, and $f_n(0) = \left[f_n(t)\right]_{t=0}$. Eqn.(3.2.1)can now be transformed using eqn.(3.2.2) provided that a set of artificial variables $i_1, i_2, \ldots, i_n$ are introduced into $y_n(<i+m>T)$ such that

$$y_n(<i+m>T) = \left[y_n(<i_1+m>T, <i_2+m>T, \ldots, <i_n+m>T)\right]_{i_1=i_2=i_3=\ldots=i_n=i} \quad 0 \leq m < 1 \tag{3.2.5}$$

and then the M.D.M.Z.T of eqn.(3.2.1) becomes

$$Y_n(m, z_1, z_2, \ldots, z_n)$$

$$= \sum_{i_1=0}^{\infty} \sum_{i_2=0}^{\infty} \cdots \sum_{i_n=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} w_n(<i_1+m-k_1>T, <i_2+m-k_2>T, \ldots, <i_n+m-k_n>T) \prod_{r=1}^{n} u(k_rT)\, z_r^{-i_r}$$

$$= W_n(m, z_1, z_2, \ldots z_n) \prod_{r=1}^{n} U_1(z_r), \quad 0 \leq m < 1 \tag{3.2.6}$$

where the range of summation is from 0 to ∞, since $w_n(\tau_1, \tau_2, \ldots, \tau_n)=0$ for all $\tau_r<0$, $1 \leqslant r \leqslant n$, and $W_n(m,z_1,z_2,\ldots,z_n)$ is the M.D.M.Z.T of the n^{th} order kernel. It may be noted that letting m= 0 in the above equation gives $Y_n(z_1,z_2,\ldots,z_n)$ of eqn.(2.2.13), which defines the response of the system at the sampling instants. The advantages of using eqn.(3.2.6) in preference to eqn.(3.2.1) are obvious.

3.2.2 Multidimensional Modified Z Transform of a Volterra Kernel

The multidimensional modified z transform $W_n(m,z_1,z_2,\ldots,z_n)$ of the system kernel is the M.D.Z.T of the kernel of the cascade combination of the system and a fictitious predictor, as shown in Fig.3.2. The M.D.L.T of the cascade combination is $W_n(s_1,s_2,\ldots,s_n) \prod_{r=1}^{n} e^{s_r mT}$ and therefore, its M.D.Z.T can be obtained by a sequential process defined by

$$W_n(m,z_1,z_2,\ldots,z_n) = \frac{1}{(2\pi j)^n} \int_{c_1-j\infty}^{c_1+j\infty} \int_{c_2-j\infty}^{c_2+j\infty} \cdots \int_{c_n-j\infty}^{c_n+j\infty} W_n(s_1,s_2,\ldots,s_n) \times \prod_{r=1}^{n} \frac{e^{s_r mT}\, ds_r}{(1-e^{s_r T} z_r^{-1})}$$

$$0 \leqslant m < 1 \qquad (3.2.7)$$

for which the procedure at each stage is given by

$$\underset{MZ\ r}{L} W_{n-r}(m,z_1,z_2,\ldots,z_r,s_{r+1},s_{r+2},\ldots,s_n)$$

$$= \frac{1}{2\pi j} \int_{c_r-j\infty}^{c_r+j\infty} \frac{\underset{MZ\ r-1}{L} W_{n-r+1}(m,z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n) e^{s_r mT}\, ds_r}{(1 - e^{s_r T} z_r^{-1})}$$

$$m,z_1,z_2,\ldots,z_{r-1},s_{r+1},s_{r+2},\ldots,s_n \text{ constant}$$

$$r= 1,2,\ldots,n \text{ and } 0 \leqslant m < 1 \qquad (3.2.8)$$

where c_r is a real number in the half plane of convergence of $\underset{MZ\ r-1}{L} W_{n-r+1}(m,z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n)$. If the function $\underset{MZ\ r-1}{L} W_{n-r+1}(m,z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n)$, $r \leqslant n$, has no branch points, then the above integral may be evaluated using the residue theorem as

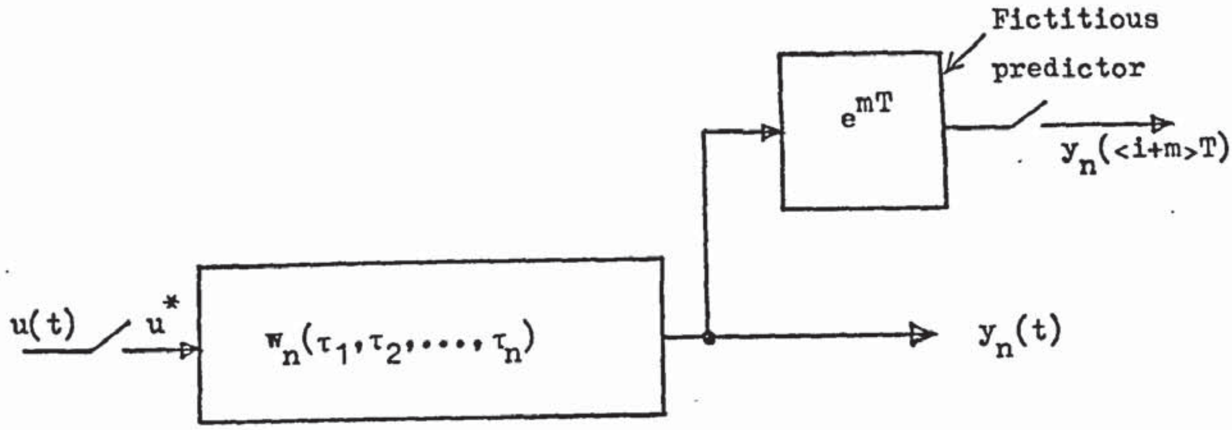


Fig.3.1 Asynchronous nonlinear sampled data system.

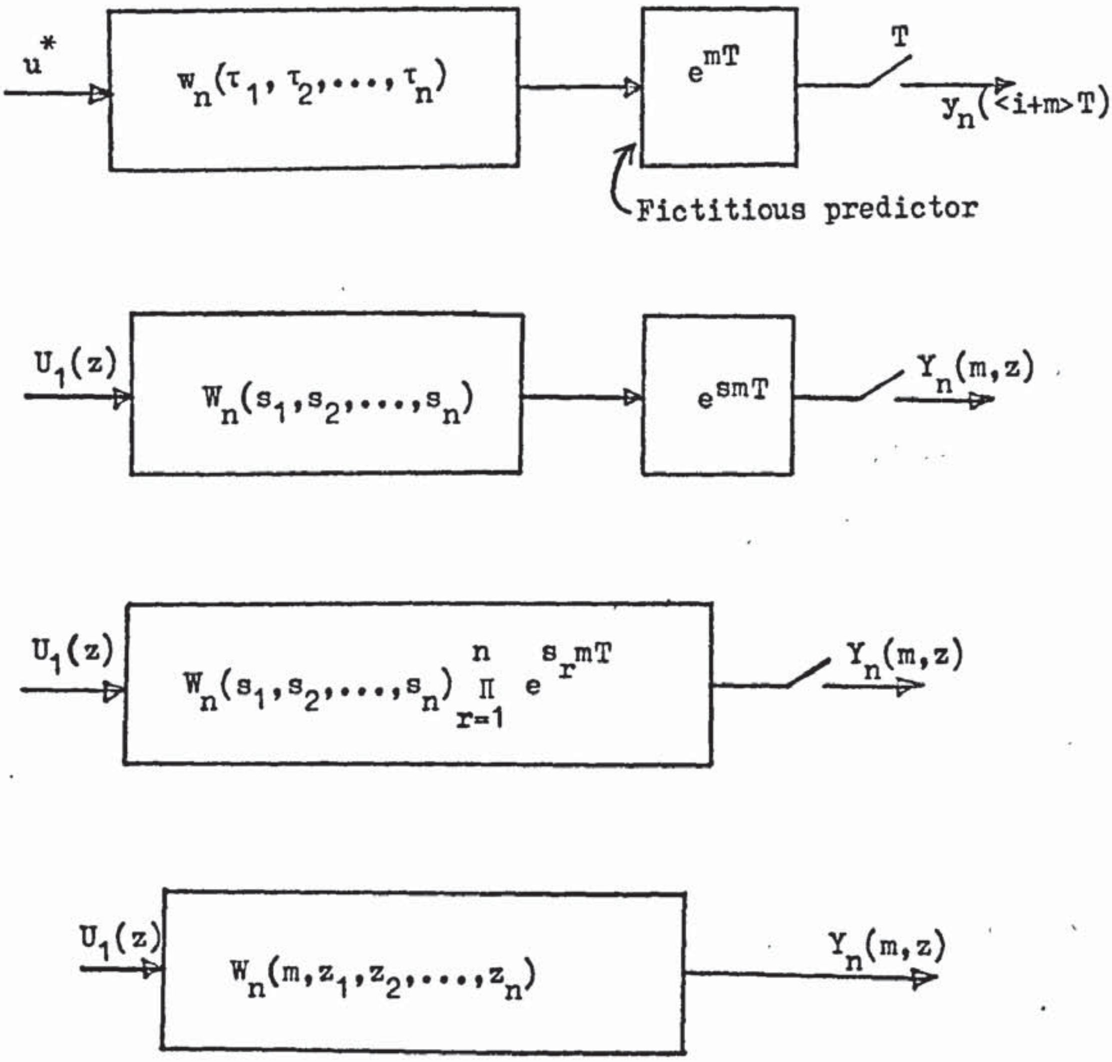


Fig.3.2 Method of obtaining multidimensional modified z transform of a system kernel.

$$
{}_{\substack{L\\MZ}}W_{n-r}{}_{r}(m,z_1,z_2,\ldots,z_r,s_{r+1},s_{r+2},\ldots,s_n)
= \sum_{\text{residues}} \frac{{}_{\substack{L\\MZ}}W_{n-r+1}{}_{r-1}(m,z_1,z_2,\ldots,z_{r-1},s_r,s_{r+1},\ldots,s_n)\, e^{s_r mT}}{(1-e^{s_rT}z_r^{-1})}
$$

$z_1,z_2,\ldots,z_{r-1},s_{r+1},s_{r+2},\ldots,s_n$ constant

$r = 1,2,\ldots,n$ and $0 \leq m < 1$ (3.2.9)

In many cases of practical interest, this procedure may be carried out by inspection using tables of related Laplace and modified z transforms.

For nonlinear systems with nonseparable kernels, the above procedure must be used and for systems with separable kernels, shown in Fig.2.1(a), it should be noted that

$$
W_n(m,z_1,z_2,\ldots,z_n) = \prod_{r=1}^{n} W_1(m,z_r)\ , \quad 0 \leq m < 1 \tag{3.2.10}
$$

where $W_1(m,z_r)$ is the M.D.M.Z.T of $w_1(\tau_r)$. But, $W_n(m,z_1,z_2,\ldots,z_n)$ for the system shown in Fig.2.1(a) may be obtained from $W_n(s_1,s_2,\ldots,s_n)$, using the above sequential process, where $W_n(s_1,s_2,\ldots,s_n)$ is given by

$$
W_n(s_1,s_2,\ldots,s_n) = \prod_{r=1}^{n} W_1(s_r) \tag{3.2.11}
$$

However, for systems with kernels separated by a sampler, as shown in Fig.2.1(b), the M.D.M.Z.T of the system kernel is given by

$$
W_n(m,z_1,z_2,\ldots,z_n) = K_1(m,\prod_{r=1}^{n} z_r) \prod_{r=1}^{n} J_1(z_r)\ , \quad 0 \leq m < 1 \tag{3.2.12}
$$

where $K_1(m,z)$ is the modified z transform and $J_1(z)$ is the z transform of $k_1(\tau)$ and $j_1(\tau)$, respectively. It is to be noted that the z transform of $k_1(\tau)$ is modified whereas the z transform of $j_1(\tau)$ is not modified. This is expected because the kernel $k_1(\tau)$ does not receive any information between the sampling instants.

3.2.3 Data-Hold Devices

For systems cascaded with data-hold devices, the method is similar to that of section 2.4, but uses the multidimensional modified z transforms $H_n(m,z_1,z_2,\ldots,z_n)$ and $G_n(m,z_1,z_2,\ldots,z_n)$ instead of the corresponding multidimensional z transforms. For instance, the M.D.M.Z.T of $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ preceded by a zero-order hold is given by

$$P_n(m,z_1,z_2,\ldots,z_n) = H_n(m,z_1,z_2,\ldots,z_n)\prod_{r=1}^{n}\left(\frac{z_r-1}{z_r}\right), \quad 0 \leq m < 1 \qquad (3.2.13)$$

where $H_n(m,z_1,z_2,\ldots,z_n)$ is obtained by applying the above sequential process to $H_n(s_1,s_2,\ldots,s_n)$ where $H_n(s_1,s_2,\ldots,s_n)$ is given by

$$H_n(s_1,s_2,\ldots,s_n) = W_n(s_1,s_2,\ldots,s_n)\prod_{r=1}^{n}\frac{1}{s_r} \qquad (3.2.14)$$

on the other hand, if $w_n(\tau_1,\tau_2,\ldots,\tau_n)$ is preceded by a first-order hold then its M.D.M.Z.T is given by

$$Q_n(m,z_1,z_2,\ldots,z_n) = G_n(m,z_1,z_2,\ldots,z_n)\prod_{r=1}^{n}\left(\frac{z_r-1}{z_r}\right)^2, \quad 0 \leq m < 1 \qquad (3.2.15)$$

where $G_n(m,z_1,z_2,\ldots,z_n)$ is obtained by applying the above sequential process to $G_n(s_1,s_2,\ldots,s_n)$, which is given by

$$G_n(s_1,s_2,\ldots,s_n) = W_n(s_1,s_2,\ldots,s_n)\prod_{r=1}^{n}\frac{(1+s_rT)}{s_r^2T} \qquad (3.2.16)$$

3.2.4 Example - Nonlinear System with 2nd Order Kernel

The use of the multidimensional modified z transform is illustrated through the analysis of the system of Fig.3.3(a). For this system, $W_2(s_1,s_2)$ is given by eqn.(2.3.10). The sampled output of this system is $y_2(\langle i+m \rangle T)$. By applying the sequential process to $W_2(s_1,s_2)$, ${}_{\substack{L\\MZ\ 1}}W_1(m,z_1,s_2)$ is obtained as

$$\underset{MZ}{L}W_1(m,z_1,s_2) = \sum_{\text{residues}} \frac{W_2(s_1,s_2)\, e^{s_1 mT}}{(1 - e^{s_1 T} z_1^{-1})}\ , \quad s_2 \text{ is constant}$$

$$= \frac{\alpha^2\beta}{(s_2+a)(s_2+b-a)} \left[\frac{e^{-amT}}{(1- e^{-aT} z_1^{-1})} - \frac{e^{-(s_2+b)mT}}{(1- e^{-(s_2+b)T} z_1^{-1})} \right]$$

$$= \frac{\alpha^2\beta}{(b-2a)} \left[\frac{e^{-amT}}{(1 - e^{-aT} z_1^{-1})} - \frac{e^{-(s_2+b)mT}}{(1 - e^{-(s_2+b)T} z_1^{-1})} \right]$$

$$\times \left[\frac{1}{(s_2+a)} - \frac{1}{(s_2+b-a)} \right]$$

$W_2(m,z_1,z_2)$ can then be obtained, from $\underset{MZ}{L}W_1(m,z_1,s_2)$ through the sequential process, as

$$W_2(m,z_1,z_2) = \sum_{\text{residues}} \frac{\underset{MZ}{L}W_1(m,z_1,s_2)\, e^{s_2 mT}}{(1 - e^{s_2 T} z_2^{-1})}\ , \quad z_1 \text{ is constant}$$

$$= \frac{\alpha^2\beta}{(b-2a)} \left[\frac{e^{-2amT}}{(1 - e^{-aT} z_1^{-1})(1 - e^{-aT} z_2^{-1})} \right.$$

$$- \frac{e^{-bmT}}{(1-e^{-aT} z_1^{-1})(1-e^{-(b-a)T} z_2^{-1})} - \frac{e^{-bmT}}{(1-e^{-bT} z_1^{-1} z_2^{-1})(1-e^{-aT} z_2^{-1})}$$

$$\left. + \frac{e^{-bmT}}{(1-e^{-bT} z_1^{-1} z_2^{-1})(1-e^{-(b-a)T} z_2^{-1})} \right]$$

$$= \frac{\alpha^2\beta\{e^{-2amT}(z_1z_2-e^{-bT})-e^{-bmT}(z_1z_2-e^{-2aT})\}\, z_1z_2}{(b-2a)(z_1z_2-e^{-bT})(z_1-e^{-aT})(z_2-e^{-aT})} \qquad (3.2.17)$$

It should be noted that by letting $m = 0$ in the above equation, $W_2(z_1,z_2)$ may be obtained, as given by eqn.(2.3.11). If the input to this system is an impulse with transform 1, then, from eqn.(3.2.6), the output is given by

$$Y_2(m,z_1,z_2) = W_2(m,z_1,z_2)\, U_1(z_1)\, U_1(z_2)$$

$$= \frac{\alpha^2\beta\{e^{-2amT}(z_1z_2-e^{-bT}) - e^{-bmT}(z_1z_2-e^{-2aT})\}\ z_1z_2}{(b-2a)(z_1z_2-e^{-bT})(z_1-e^{-aT})(z_2-e^{-aT})}$$

If, however, the system of Fig.3.3(a) is cascaded with a zero-order hold, as shown in Fig.3.3(b), then $H_2(s_1,s_2)$ for the system is given by eqn.(2.4.3), from which ${}_{\substack{L\\MZ}}H_{\substack{1\\1}}(m,z_1,z_2)$ is obtained, through the sequential process, as

$${}_{\substack{L\\MZ}}H_{\substack{1\\1}}(m,z_1,s_2) = \sum_{\text{residues}} \frac{H_2(s_1,s_2)e^{s_1mT}}{(1-e^{s_1T}z_1^{-1})}$$

$$= \alpha^2\beta\left[\frac{z_1}{a(z_1-1)s_2(s_2+a)(s_2+b)} - \frac{z_1\ e^{-amT}}{a(z_1-e^{-aT})s_2(s_2+a)(s_2+b-a)} + \frac{z_1z_2e^{-bmT}\ e^{-s_2mT}}{(z_1z_2-e^{-bT})s_2(s_2+a)(s_2+b)(s_2+b-a)}\right]$$

Then, $H_2(m,z_1,z_2)$ may be obtained by applying the sequential process to ${}_{\substack{L\\MZ}}H_{\substack{1\\1}}(m,z_1,s_2)$, which gives

$$H_2(m,z_1,z_2) = \sum_{\text{residues}} {}_{\substack{L\\MZ}}H_{\substack{1\\1}}(m,z_1,s_2)\ \frac{e^{s_2mT}}{(1-e^{s_2T}z_2^{-1})}$$

$$= \alpha^2\beta z_1z_2\frac{\begin{aligned}\Big[&(b-a)(b-2a)(z_1-e^{-aT})(z_2-e^{-aT})(z_1z_2-e^{-bT})\\ &-b(b-2a)e^{-amT}(z_1z_2-e^{-bT})\{(z_1-1)(z_2-e^{-aT})+(z_1-e^{-aT})(z_2-1)\}\\ &+a(b-2a)e^{-bmT}(z_1z_2-1)(z_1-e^{-aT})(z_2-e^{-aT})\\ &+b(b-a)e^{-2amT}(z_1-1)(z_2-1)(z_1z_2-e^{-bT})\\ &-abe^{-bmT}(z_1-1)(z_2-1)(z_1z_2-e^{-2aT})\Big]\end{aligned}}{a^2b(b-a)(b-2a)(z_1-1)(z_2-1)(z_1-e^{-aT})(z_2-e^{-aT})(z_1z_2-e^{-bT})} \tag{3.2.18}$$

Thus, the M.D.M.Z.T of the system preceded by a zero-order hold is given by

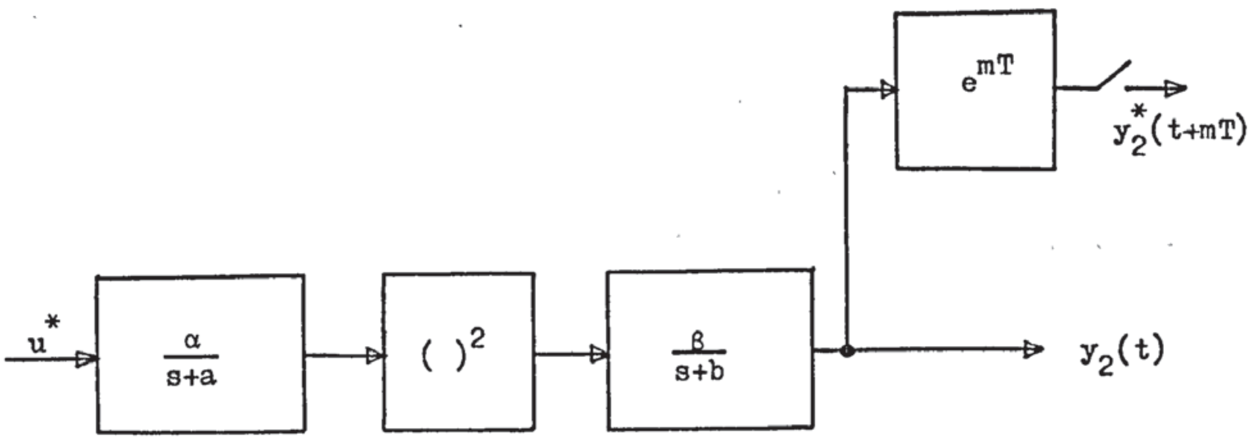


Fig.3.3(a) Nonlinear system with 2nd-order kernel.

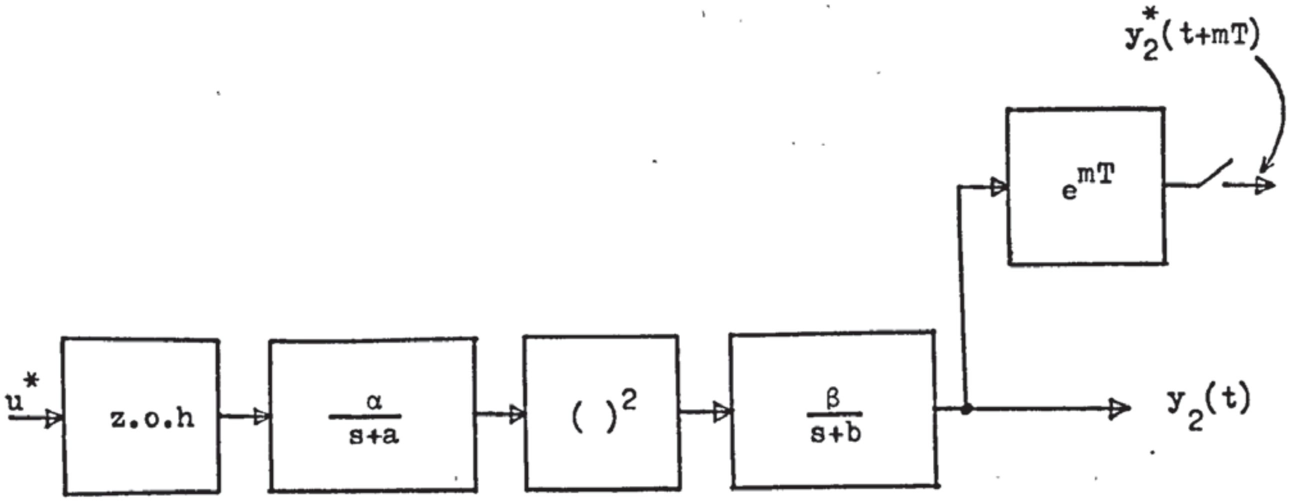


Fig.3.3(b) 2nd-order nonlinear system preceded by zerO-order hold.

$$P_2(m,z_1,z_2) = H_2(m,z_1,z_2) \prod_{r=1}^{2} \left(\frac{z_r-1}{z_r}\right)$$

$$= \alpha^2\beta \frac{\begin{aligned}\Big[&(b-a)(z_1z_2-e^{-bT})\{(b-2a)(z_1-e^{-aT})(z_2-e^{-aT}) \\ &+be^{-2amT}(z_1-1)(z_2-1)\}-b(b-2a)(z_1z_2-e^{-bT})\{(z_1-1)(z_2-e^{-aT}) \\ &+(z_1-e^{-aT})(z_2-1)\}+a(b-2a)e^{-bmT}(z_1z_2-1)(z_1-e^{-aT})(z_2-e^{-aT}) \\ &-abe^{-bmT}(z_1-1)(z_2-1)(z_1z_2-e^{-2aT})\Big]\end{aligned}}{a^2b(b-a)(z_1-e^{-aT})(z_2-e^{-aT})(z_1z_2-e^{-bT})} \tag{3.2.19}$$

which is the required result.

The output of this system for a unit impulse is given by

$$Y_2(m,z_1,z_2) = P_2(m,z_1,z_2) \tag{3.2.20}$$

The inverse of the output modified z transform $Y_2(m,z_1,z_2)$ yields the sampled output $y_2(<i_1+m>T, <i_2+m>T)$ and by equating $i_1=i_2=i$, $y_2(<i+m>T)$ can be obtained, where $0<m<1$. But, this method of determining $y_2(<i+m>T)$ leads to a cumbersome process and hence an association-of-variables procedure for multidimensional modified z transform is developed in the next section, by which the complex variables z_1,z_2 in $Y_2(m,z_1,z_2)$ may be associated to yield $Y_1(m,z)$ and inverting $Y_1(m,z)$ yields $y_2(<i+m>T)$, as required.

3.3 Association Of Variables in Multidimensional Modified Z Transforms

A sequential process, for the association-of-variables in M.D.M.Z.T, may be developed in a way similar to the one developed for M.D.Z.T. Alternatively, some theorems are developed here for multidimensional modified z transforms, analogous to those developed for M.D.L.T[85,86], by which the z variables in M.D.M.Z.T may be associated. These theorems are also found to be useful in obtaining the discrete state transition equation of the sampled-data nonlinear system(Chapter 7), from the solution of its discrete state equation.

3.3.1 Theorems of Association Of Variables

Theorem 1: Real Translation Theorem(Backward shifting theorem)

If, $Y_n(m,z_1,z_2,\dots,z_n)$ may be expressed in the form

$$Y_n(m,z_1,z_2,\dots,z_n) = \left(\prod_{p=1}^{r} z_p^{-k}\right) F_n(m,z_1,z_2,\dots,z_n)$$
$$0 \leqslant m < 1 \tag{3.3.1}$$

then the real translation theorem(backward shifting theorem) can be used to obtain the associated transform. The backward shifting theorem states that, if $F_1(m,z)$ is the associated transform of $F_n(m,z_1,z_2,\dots,z_n)$, then

$$Y_1(m,z) = z^{-k} F_1(m,z) , \qquad 0 < m < 1 \tag{3.3.2}$$

where $Y_1(m,z)$ is the required associated transform of $Y_n(m,z_1,z_2,\dots,z_n)$. The associated transform $F_1(m,z)$ may be obtained from $F_n(m,z_1,z_2,\dots,z_n)$ by using the theorems described in this section.

Proof: By definition, the multidimensional inverse modified z transformation of $Y_n(m,z_1,z_2,\dots,z_n)$ is given by

$$y_n(\langle i_1+m\rangle T,\langle i_2+m\rangle T,\dots,\langle i_n+m\rangle T)$$
$$= \frac{1}{(2\pi j)^n} \oint_{C_1}\oint_{C_2}\dots\oint_{C_n} Y_n(m,z_1,z_2,\dots,z_n) \prod_{p=1}^{n} z_p^{i_p-1}\, dz_p$$
$$0 < m < 1 \tag{3.3.3}$$

Setting $i_1=i_2=\dots=i_n=i$, and substituting eqn.(3.3.1) into eqn.(3.3.3) gives

$$y_n(\langle i+m\rangle T) = \frac{1}{(2\pi j)^n} \oint_{C_1}\oint_{C_2}\dots\oint_{C_n} F_n(m,z_1,z_2,\dots,z_n) \prod_{p=1}^{n} z_p^{i-k-1}\, dz_p$$
$$= \left[f_n(\langle i_1-k+m\rangle T,\langle i_2-k+m\rangle T,\dots,\langle i_n-k+m\rangle T)\right]_{i_1=i_2=\dots=i_n=i}$$
$$= f_n(\langle i-k+m\rangle T) \tag{3.3.4}$$

where $0 \leqslant m < 1$ and i and k are integers. Taking a one-dimensional modified z transform on both sides of eqn.(3.3.4) gives

$$Y_1(m,z) = z^{-k}\, F_1(m,z)\ , \qquad 0 \leqslant m < 1 \tag{3.3.5}$$

where $F_1(m,z)$ and $Y_1(m,z)$ are modified z transforms of $f_n(<i+m>\,T)$ and $y_n(<i+m>T)$, respectively.

Corollary 1.1: Forward shifting theorem

The forward shifting theorem states that if, $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form as

$$Y_n(m,z_1,z_2,\ldots,z_n) = \left(\prod_{p=1}^{r} z_p^k\right) F_n(m,z_1,z_2,\ldots,z_n) \tag{3.3.6}$$

then $Y_1(m,z)$ is given by

$$Y_1(m,z) = z^k \left[F_1(m,z) - \sum_{l=0}^{k-1} f_n(<l+m>\,T)\, z^{-l}\right], \tag{3.3.7}$$

where $f_n(<i+m>T)$ is the inverse transform of $F_1(m,z)$.

Proof: Setting $i_1=i_2=\ldots=i_n=i$ and substituting eqn.(3.3.6) into eqn.(3.3.3) yields

$$\begin{aligned} y_n(<i+m>\,T) &= \frac{1}{(2\pi j)^n} \oint_{C_1}\oint_{C_2}\ldots\oint_{C_n} F_n(m,z_1,z_2,\ldots,z_n) \prod_{p=1}^{n} z_p^{i+k-1}\, dz_p \\ &= f_n(<i+k+m>\,T)\ , \quad 0 \leqslant m < 1 \end{aligned} \tag{3.3.8}$$

Taking the modified z transform on either side of eqn.(3.3.8) yields[83]

$$Y_1(m,z) = z^k \left[F_1(m,z) - \sum_{i=0}^{k-1} f_n(<i+m>\,T)\, z^{-i}\right]\ , \quad 0 \leqslant m < 1 \tag{3.3.9}$$

Theorem 2: Complex Translation Theorem

If $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form as

$$Y_n(m,z_1,z_2,\ldots,z_n) = \frac{z_k\, e^{\mp amT}}{(z_k - e^{\mp aT})} F_{n-1}(m,z_1,z_2,\ldots,z_{k-1},z_{k+1},\ldots,z_n)$$

$$k < n \text{ and } 0 \leqslant m < 1 \tag{3.3.10}$$

then the complex translation theorem states that $Y_1(m,z)$ is given by

$$Y_1(m,z) = e^{\mp amT} F_1(m,ze^{\pm aT}) \ , \ 0 < m < 1 \qquad (3.3.11)$$

where $F_1(m,z)$ is the associated transform of $F_{n-1}(m,z_1,z_2,\ldots,z_{k-1},z_{k+1},\ldots,z_n)$.

<u>Proof</u> : Setting $i_1=i_2=\ldots=i_n=i$, and substituting eqn.(3.3.10) into eqn.(3.3.3) gives

$$y_n(<i+m>T) = \{\frac{1}{2\pi j}\oint_{C_k} \frac{z_k e^{\mp amT}}{(z_k - e^{\mp aT})} z_k^{i-1}\, dz_k\}$$

$$\times\{\frac{1}{(2\pi j)^{n-1}} \oint_{C_1}\ldots\oint_{C_{k-1}} \oint_{C_{k+1}}\ldots\oint_{C_n} F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \times \prod_{\substack{p=1\\p\neq k}}^{n} z_p^{i-1}\, dz_p\}$$

$$= e^{\mp a(i+m)T}\Big[f_{n-1}(<i_1+m>T,\ldots,<i_{k-1}+m>T,<i_{k+1}+m>T,\ldots,<i_n+m>T)\Big]_{\substack{i_1=i_2=\ldots=i_{k-1}=i_{k+1}=\ldots=i_n=i\\ k<n}} ,$$

$$= e^{\mp a(i+m)T} f_{n-1}(<i+m>T) \quad , \quad 0 < m < 1 \qquad (3.3.12)$$

Taking the modified z transform on both sides of the above equation yields

$$Y_1(m,z) = e^{\mp amT} F_1(m,ze^{\pm aT}) \quad , \quad 0 < m < 1 \qquad (3.3.13)$$

where $F_1(m,z)$ is the modified z transform of $f_{n-1}(<i+m>T)$.

<u>Corollary 2.1</u> : If, $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form

$$Y_n(m,z_1,z_2,\ldots,z_n) = \frac{e^{\mp amT}}{(z_k - e^{\mp aT})} F_{n-1}(m,z_1,z_2,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \qquad (3.3.14)$$

then $Y_1(m,z)$ is given by

$$Y_1(m,z) = e^{\mp a(m-1)T}\{ F_1(m,ze^{\pm aT}) - \lim_{z\to\infty} F_1(m,z)\} \quad , \quad 0 < m < 1 \quad (3.3.15)$$

<u>Proof</u> : Setting $i_1= i_2=\ldots=i_n=i$ and substituting eqn.(3.3.14) in eqn.(3.3.3) gives

$$y_n(<i+m>T) = \{\frac{1}{2\pi j} \oint_{C_k} \frac{e^{\mp amT}}{(z_k - e^{\mp aT})} z_k^{i-1} dz_k\}$$

$$\times \{\frac{1}{(2\pi j)^{n-1}} \oint_{C_1} \cdots \oint_{C_{k-1}} \oint_{C_{k+1}} \cdots \oint_{C_n} F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n)$$

$$\times \prod_{\substack{p=1 \\ p\neq k}}^{n} z_p^{i-1} dz_p\}$$

$$= e^{\mp a(i+m-1)T} f_{n-1}(<i+m>T) \quad ; \quad 0 \leq m < 1; \ i \geq 1 \qquad (3.3.16)$$

Taking a one dimensional modified z transform on either side of eqn. (3.3.16) gives

$$Y_1(m,z) = e^{\mp a(m-1)T}\{ F_1(m,ze^{\pm aT}) - f_1(mT)\}$$

$$= e^{\mp a(m-1)T}\{ F_1(m,ze^{\pm aT}) - \lim_{z\to\infty} F_1(m,z)\}$$

$$0 \leq m < 1 \qquad (3.3.17)$$

where the initial value theorem (Appendix A.3.1) has been used to evaluate $f_1(mT)$.

Corollary 2.2 If $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form as

$$Y_n(m,z_1,z_2,\ldots,z_n) = \frac{z_k}{(z_k-1)} F_{n-1}(m,z_1,z_2,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \qquad (3.3.18)$$

then $Y_1(m,z)$ is given by

$$Y_1(m,z) = F_1(m,z) \qquad (3.3.19)$$

This follows immediately, if it is noted that the inverse z transform of $(\frac{z}{z-1})$ is unity. This result may also be obtained from eqn.(3.3.13) by putting $a = 0$.

Theorem 3 Complex Convolution Theorem

If $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form

$$Y_n(m,z_1,z_2,\ldots,z_n) = F_k(m,z_1,z_2,\ldots,z_k)\, S_{n-k}(m,z_{k+1},z_{k+2},\ldots,z_n)$$

$$k < n \qquad (3.3.20)$$

then the complex convolution theorem may be used to obtain $Y_1(m,z)$. The theorem states that, if $F_1(m,z)$ and $S_1(m,z)$ are the associated transforms of $F_k(m,z_1,z_2,\ldots,z_k)$ and $S_{n-k}(m,z_{k+1},\ldots,z_n)$, respectively, then $Y_1(m,z)$ is given by

$$Y_1(m,z) = F_1(m,z) \circledast S_1(m,z) \quad , \quad 0 \leq m < 1 \qquad (3.3.21)$$

where the symbol $\circledast$ denotes complex convolution.

Proof : Setting $i_1=i_2=\ldots=i_n=i$ and substituting eqn.(3.3.20) into eqn. (3.3.3) gives

$$\begin{aligned} y_n(<i+m>T) &= \frac{1}{(2\pi j)^k} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_k} F_k(m,z_1,z_2,\ldots,z_k) \prod_{p=1}^{k} z_p^{i-1}\, dz_p \\ &\quad \times \frac{1}{(2\pi j)^{n-k}} \oint_{C_{k+1}} \cdots \oint_{C_n} S_{n-k}(m,z_{k+1},\ldots,z_n) \prod_{q=(k+1)}^{n} z_q^{i-1}\, dz_q \\ &= \Big[f_k(<i_1+m>T,<i_2+m>T,\ldots,<i_k+m>T)\Big]_{i_1=i_2=\ldots=i_k=i} \\ &\quad \times \Big[s_{n-k}(<i_{k+1}+m>T,\ldots,<i_n+m>T)\Big]_{i_{k+1}=\ldots=i_n=i} \\ &= f_k(<i+m>T)s_{n-k}(<i+m>T) \quad , \quad 0 \leq m < 1 \qquad (3.3.22) \end{aligned}$$

Taking the modified z transform on both sides of the above equation yields,

$$\begin{aligned} Y_1(m,z) &= \frac{1}{2\pi j} \oint_C \frac{F_1(m,\frac{z}{w})\; S_1(m,w)}{w}\, dw \\ &= \sum_{\substack{\text{residues} \\ \text{within } C}} \frac{F_1(m,\frac{z}{w})\; S_1(m,w)}{w} \quad , \quad 0 \leq m < 1 \qquad (3.3.23) \end{aligned}$$

where the contour C is a circle in the region $\sigma_1 < |w| < |z|/\sigma_2$ and $|z| > \max(\sigma_1,\sigma_2,\sigma_1 \cdot \sigma_2)$, σ_1 is the radius of convergence of $S_1(m,w)$, σ_2 is the radius of convergence of $F_1(m,w)$ for $0 \leq m < 1$, and $F_1(m,z)$ and $S_1(m,z)$ are modified z transforms of $f_k(<i+m>T)$ and $s_{n-k}(<i+m>T)$, respectively. The above equation may be evaluated by using residue theorem.

Theorem 4 : Real Discrete Convolution Theorem

If $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form as

$$Y_n(m,z_1,z_2,\dots,z_n) = S_1(m,\prod_{r=1}^{n} z_r)\; F_n(z_1,z_2,\dots,z_n) \quad , \quad 0 \leqslant m < 1 \quad (3.3.24)$$

then the associated transform $Y_1(m,z)$ may be obtained by using the real discrete convolution theorem. The theorem states that if $F_1(z)$ is the associated transform of $F_n(z_1,z_2,\dots,z_n)$, then $Y_1(m,z)$ is given by

$$Y_1(m,z) = S_1(m,z)\; F_1(z) \quad , \quad 0 \leqslant m < 1 \qquad (3.3.25)$$

Proof : Subsitituting eqn.(3.3.24) into eqn.(3.3.3) for $Y_n(m,z_1,z_2,\dots,z_n)$, yields

$$y_n(<i_1+m>T, <i_2+m>T,\dots,<i_n+m>T)$$
$$= \frac{1}{(2\pi j)^n} \oint_{C_1}\oint_{C_2}\dots\oint_{C_n} S_1(m,\prod_{p=1}^{n} z_p)\; F_n(z_1,z_2,\dots,z_n) \prod_{p=1}^{n} z_p^{i_p-1}\, dz_p$$
$$= \sum_{k=0}^{\min(i_1,i_2,\dots,i_n)} s_1(<k+m>T) f_n(<i_1-k>T,<i_2-k>T,\dots,<i_n+k>T)$$
$$0 \leqslant m < 1 \qquad (3.3.26)$$

Setting $i_1=i_2=\dots=i_n=i$, in the above equation, gives

$$y_n(<i+m>T) = \sum_{k=0}^{i} s_1(<k+m>T)\; f_n(<i-k>T) \quad , \quad 0 \leqslant m < 1 \qquad (3.3.27)$$

Taking the modified z transform on either side of the above equation, yields

$$Y_1(m,z) = S_1(m,z)\; F_1(z) \quad , \quad 0 \leqslant m < 1 \qquad (3.3.28)$$

Theorem 5 : Differentiation Theorem

If $Y_n(m,z_1,z_2,\dots,z_n)$ may be expressed in the form as

$$Y_n(m,z_1,z_2,\dots,z_n) = \frac{(-1)^l}{l!}\; \frac{d^l}{da^l} \left(\frac{z_k e^{-amT}}{z_k - e^{-aT}}\right) F_{n-1}(m,z_1,\dots,z_{k-1},z_{k+1},\dots,z_n)$$
$$(3.3.29)$$

then, the differentiation theorem may be used to obtain $Y_1(m,z)$. The differentiation theorem states that if $F_1(m,z)$ is the associated transform of $F_{n-1}(m,z_1,z_2,\dots,z_{k-1},z_{k+1},\dots,z_n)$, then $Y_1(m,z)$ is given by

$$Y_1(m,z) = \frac{(-1)^l}{l!}\frac{d^l}{da^l}\left[e^{-amT}F_1(m,ze^{aT})\right], \quad 0\leqslant m<1 \qquad (3.3.30)$$

Proof : Setting $i_1=i_2=\ldots=i_k=\ldots=i_n=i$ and substituting eqn.(3.3.29) into eqn.(3.3.3) gives

$$y_n(<i+m>T) = \left[\frac{1}{2\pi j}\oint_{C_k}\frac{(-1)^l}{l}\frac{d^l}{da^l}\left(\frac{z_k e^{-amT}}{z-e^{-aT}}\right) z_k^{i-1}\,dz_k\right]$$

$$\times\left[\frac{1}{(2\pi j)^{n-1}}\oint_{C_1}\oint_{C_2}\ldots\oint_{C_{k-1}}\oint_{C_{k+1}}\ldots\oint_{C_n} F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n)\times\prod_{\substack{p=1\\p\neq k}}^{n} z_p^{i-1}dz_p\right]$$

$$= \frac{(iT)^l e^{-a(i+m)T}}{l!}\left[f_{n-1}(<i_1+m>T,\ldots,<i_{k-1}+m>T,<i_{k+1}+m>T,\ldots,<i_n+m>T)\right]_{i_1=i_2=\ldots=i_{k-1}=i_{k+1}=\ldots=i_n=i}$$

$$= \frac{(iT)^l}{l!}\,e^{-a(i+m)T}f_{n-1}(<i+m>T) \qquad (3.3.31)$$

Taking the z transform on either side, and using the complex translation theorem, yields

$$Y_1(m,z) = \frac{(-1)^l}{l!}\frac{d^l}{da^l}\{e^{-amT}F_1(m,ze^{aT})\} \qquad (3.3.32)$$

Corollary 5.1 : If, however, $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form as

$$Y_n(m,z_1,z_2,\ldots,z_n) = \lim_{a\to 0}\left\{\frac{(-1)^l}{l!}\frac{d^l}{da^l}\left(\frac{z_k e^{-amT}}{z_k-e^{-aT}}\right)\right\}F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \qquad (3.3.33)$$

then $Y_1(m,z)$ is given by

$$Y_1(m,z) = \lim_{a\to 0}\left[\frac{(-1)^l}{l!}\frac{d^l}{da^l}\{e^{-amT}F_1(m,ze^{aT})\}\right] \qquad (3.3.34)$$

This is easily proved by using the above theorem and corollary 2.2.

Theorem 6 : Theorems Related to Sinusoidal functions :

If $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form as

$$Y_n(m,z_1,z_2,\ldots,z_n) = \left\{\frac{z_k^2\sin amT + z_k\sin a(1-m)T}{z_k^2-2z_k\cos aT+1}\right\}F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n)$$

$$k < n \quad \text{and} \quad 0 \leqslant m < 1 \tag{3.3.35}$$

then the associated transform $Y_1(m,z)$ is given by

$$Y_1(m,z) = \frac{1}{2j}\{e^{jamT} F_1(m,ze^{-jaT}) - e^{-jamT} F_1(m,ze^{jaT})\} \tag{3.3.36}$$

where $j=\sqrt{-1}$. The transform of the form given by eqn.(3.3.35) occur in systems having oscillating type kernels or in systems with sinusoidal input signals. The theorem may be proved in two ways and both the proofs are given here.

<u>Proof</u> : (a). First, eqn.(3.3.35) may be rewritten as

$$Y_n(m,z_1,z_2,\ldots,z_n) = \frac{1}{2j}\left[\frac{z_k e^{jamT}}{(z_k-e^{jaT})} - \frac{z_k e^{-jamT}}{(z_k-e^{-jaT})}\right] F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \tag{3.3.37}$$

Then, letting $i_1=i_2=\ldots=i_k=\ldots=i_n=i$ and substituting eqn.(3.3.37) into eqn.(3.3.3) yields

$$\begin{aligned} y_n(<i+m>T) &= \frac{1}{2j}\left[\{\frac{1}{2\pi j}\oint_{C_k}\frac{z_k e^{jamT}}{(z_k-e^{jaT})} z_k^{i-1}dz_k\} - \{\frac{1}{2\pi j}\oint_{C_k}\frac{z_k e^{-jamT}}{(z_k-e^{-jaT})} z_k^{i-1}dz_k\}\right] \\ &\quad \times \left[\frac{1}{(2\pi j)^{n-1}}\oint_{C_1}\cdots\oint_{C_{k-1}}\oint_{C_{k+1}}\cdots\oint_{C_n} F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \right. \\ &\qquad \left. \times \prod_{\substack{p=1 \\ p\neq k}}^{n} z_p^{i-1}\,dz_p\right] \\ &= \frac{\{e^{ja(i+m)T} - e^{-ja(i+m)T}\}}{2j}\Big[f_{n-1}(<i_1+m>T,\ldots,<i_{k-1}+m>T,<i_{k+1}+m>T, \\ &\qquad \ldots, <i_n+m>T)\Big]_{i_1=\ldots=i_{k-1}=i_{k+1}=\ldots=i_n=i} \\ &= \frac{1}{2j}\{e^{ja(i+m)T} f_{n-1}(<i+m>T) - e^{-ja(i+m)T} f_{n-1}(<i+m>T)\} \end{aligned} \tag{3.3.38}$$

using the complex translation theorem, the modified z transform of eqn. (3.3.38) gives $Y_1(m,z)$ as

$$Y_1(m,z) = \frac{1}{2j}\{e^{jamT} F_1(m,ze^{-jaT}) - e^{-jamT} F_1(m,ze^{jaT})\} \quad ,0 \leqslant m < 1 \tag{3.3.39}$$

(b). Alternatively, substituting eqn.(3.3.35) into eqn.(3.3.3) and letting $i_1 = i_2 = \ldots = i_n = i$, gives

$$y_n(<i+m>T) = \left[\frac{1}{2\pi j}\oint_{C_k}\left\{\frac{z_k^2 \sin amT + z_k \sin a(1-m)T}{z_k^2 - 2z_k\cos aT + 1}\right\} z_k^{i-1}\, dz_k\right]$$

$$\times\left[\frac{1}{(2\pi j)^{n-1}}\oint_{C_1}\cdots\oint_{C_{k-1}}\oint_{C_{k+1}}\cdots\oint_{C_n} F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \times \prod_{\substack{p=1\\p\neq k}}^{n} z_p^{i-1}\, dz_p\right]$$

$$= \sin a(i+m)T\, f_{n-1}(<i+m>T)$$

$$= \frac{1}{2j}\left[e^{ja(i+m)T} f_{n-1}(<i+m>T) - e^{-ja(i+m)T} f_{n-1}(<i+m>T)\right] \qquad (3.3.40)$$

which is same as eqn.(3.3.38). Thus, taking modified z transform of eqn.(3.3.40) gives eqn.(3.3.39), which proves the theorem. On similar lines, the following may be easily proved.

Corollary 6.1 : If $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form

$$Y_n(m,z_1,z_2,\ldots,z_n) = \left\{\frac{z_k^2\cos amT - z_k\cos(1-m)aT}{z_k^2 - 2z_k\cos aT + 1}\right\} F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n)$$

$$0 \leq m < 1 \qquad (3.3.41)$$

then $Y_1(m,z)$ is given by the modified z transform of

$$y_n(<i+m>T) = \cos a(i+m)T\, f_{n-1}(<i+m>T) \quad , \quad 0 \leq m < 1 \qquad (3.3.42)$$

which gives $Y_1(m,z)$ as

$$Y_1(m,z) = \frac{1}{2}\left[e^{jamT} F_1(m,ze^{-jaT}) + e^{-jamT} F_1(m,ze^{jaT})\right], 0 \leq m < 1 \qquad (3.3.43)$$

Corollary 6.2 : If $Y_n(m,z_1,z_2,\ldots,z_n)$ may be expressed in the form

$$Y_n(m,z_1,z_2,\ldots,z_n) = \frac{z_k^2\sinh amT + z_k\sinh(1-m)aT}{z_k^2 - 2z_k\cosh aT + 1}\, F_{n-1}(m,z_1,\ldots,z_{k-1},z_{k+1},\ldots,z_n) \qquad (3.3.44)$$

then $Y_1(m,z)$ is given by the modified z transform of

$$y_n(<i+m>T) = \sinh a(i+m)T\, f_{n-1}(<i+m>T) \quad , \quad 0 \leq m < 1 \qquad (3.3.45)$$

which gives $Y_1(m,z)$ as

$$Y_1(m,z) = \frac{1}{2}\left[e^{amT} F_1(m,ze^{-aT}) - e^{-amT} F_1(m,ze^{aT})\right] \quad , \quad 0 \leq m < 1 \qquad (3.3.46)$$

Corollary 6.3 : If $Y_n(m,z_1,z_2,\dots,z_n)$ may be expressed in the form

$$Y_n(m,z_1,z_2,\dots,z_n) = \frac{z_k^2\cosh amT - z_k\cosh(1-m)aT}{z_k^2 - 2z_k\cosh aT + 1} F_{n-1}(m,z_1,\dots,z_{k-1},z_{k+1},\dots,z_n) \tag{3.3.47}$$

then $Y_1(m,z)$ is given by the modified z transform of

$$y_n(<i+m>T) = \cosh a(i+m)T\, f_{n-1}(<i+m>T) \quad , \quad 0 \leq m < 1$$

which gives $Y_1(m,z)$ as

$$Y_1(m,z) = \frac{1}{2}\left[e^{amT} F_1(m,ze^{-aT}) + e^{-amT} F_1(m,ze^{aT})\right] \quad , \quad 0 \leq m < 1 \tag{3.3.48}$$

The associated transform $F_1(m,z)$ may be obtained by using one or more of the above theorems.

3.3.2 Example - Nonlinear System with 3rd Order Kernel

To illustrate the use of the theorems for the association-of-variables, consider the nonlinear system preceded by a zero-order hold as shown in Fig.3.4, for which the M.D.L.T of the cascade is given by

$$H_3(s_1,s_2,s_3) = W_3(s_1,s_2,s_3)\prod_{r=1}^{3}\frac{1}{s_r} = \frac{1}{s_1s_2s_3(s_1+a)(s_2+s_3+a)(s_1+s_2+s_3+a)}$$

where $W_3(s_1,s_2,s_3)$ is given by

$$W_3(s_1,s_2,s_3) = \frac{1}{(s_1+a)(s_2+s_3+a)(s_1+s_2+s_3+a)} \tag{3.3.49}$$

The M.D.M.Z.T of the cascade is then given by

$$\begin{aligned}P_3(m,z_1,z_2,z_3) &= H_3(m,z_1,z_2,z_3)\prod_{r=1}^{3}\left(\frac{z_r-1}{z_r}\right)\\
&= \frac{\left[\begin{aligned}&(z_1-e^{-aT})(z_2z_3-e^{-aT})(z_1z_2z_3-e^{-aT})\\ &- e^{-2amT}(z_1-1)(z_2z_3-1)(z_1z_2z_3-e^{-aT})\\ &- aTe^{-amT}\{mz_1z_2z_3+e^{-aT}(1-m)\}\{(z_1-e^{-aT})(z_2z_3-1)\\ &+(z_1-1)(z_2z_3-e^{-aT})\} - e^{-amT}(1-e^{-aT})\{z_2z_3(z_1-1)(z_1-e^{-aT})\\ &+ z_1(z_2z_3-1)(z_2z_3-e^{-aT})\}\end{aligned}\right]}{a^3(z_1-e^{-aT})(z_2z_3-e^{-aT})(z_1z_2z_3-e^{-aT})}\end{aligned} \tag{3.3.50}$$

where $H_3(m,z_1,z_2,z_3)$ is obtained from $H_3(s_1,s_2,s_3)$ by using the sequential process described in section 3.2. If the input to this system is a sampled step with z transform $U_1(z) = \frac{z}{z-1}$, then the output transform is given by

$$Y_3(m,z_1,z_2,z_3) = P_3(m,z_1,z_2,z_3) \prod_{r=1}^{3} \left(\frac{z_r}{z_r-1}\right)$$

$$= z_1 z_2 z_3 \, K_3(m,z_1,z_2,z_3) \qquad (3.3.51)$$

where

$$K_3(m,z_1,z_2,z_3) = \frac{1}{a^3}\Bigg[A_3(z_1,z_2,z_3) - \frac{1}{(z_1-e^{-aT})} B_2(m,z_2,z_3) - aT\,F_1(m,z_1z_2z_3)\left\{\frac{1}{(z_1-1)} B_2(z_2,z_3) + \frac{C_2(z_2,z_3)}{(z_1-e^{-aT})}\right\} - (1-e^{-aT})R_1(m,z_1z_2z_3)\{D_3(z_1,z_2,z_3)+S_1(z_1)J_2(z_2,z_3)\}\Bigg] \qquad (3.3.52)$$

$$A_3(z_1,z_2,z_3) = \frac{1}{(z_1-1)(z_2-1)(z_3-1)}, \quad C_2(z_2,z_3) = \frac{1}{(z_2-1)(z_3-1)},$$

$$B_2(m,z_2,z_3) = \frac{e^{-2amT}(z_2z_3-1)}{(z_2-1)(z_3-1)(z_2z_3-e^{-aT})}, \quad B_2(z_2,z_3) = B_2(m,z_2,z_3)\Big|_{m=0},$$

$$D_3(z_1,z_2,z_3) = \frac{z_2z_3}{(z_2-1)(z_3-1)(z_2z_3-e^{-aT})}, \quad S_1(z_1) = \frac{z_1}{(z_1-1)(z_1-e^{-aT})},$$

$$R_1(m,z_1z_2z_3) = \frac{e^{-amT}}{(z_1z_2z_3-e^{-aT})}, \quad J_2(z_2,z_3) = \frac{(z_2z_3-1)}{(z_2-1)(z_3-1)} \quad \text{and}$$

$$F_1(m,z_1z_2z_3) = \frac{e^{-amT}\{mz_1z_2z_3+e^{-aT}(1-m)\}}{(z_1z_2z_3-e^{-aT})}. \qquad (3.3.53)$$

The associated transform $Y_1(m,z)$ is obtained, from eqn.(3.3.51) by using the forward shifting theorem, as

$$Y_1(m,z) = z\,K_1(m,z) \qquad (3.3.54)$$

where $K_1(m,z)$ is the associated transform of $K_3(m,z_1,z_2,z_3)$, which is to be determined. In order to determine $K_1(m,z)$, it is necessary to determine the associated transform of eqns.(3.3.53). The associated transforms of eqns.(3.3.53) are obtained, using the theorems developed in section 3.3.1, as

$$A_1(z) = \frac{1}{z-1} , \; B_1(m,z) = \frac{e^{-2amT}}{(z-e^{-aT})} , \; B_1(z) = B_1(m,z)\Big|_{m=0} = \frac{1}{(z-e^{-aT})} ,$$

$$F_1(m,z) = \frac{e^{-amT}\{mz + e^{-aT}(1-m)\}}{(z - e^{-aT})} , \; C_1(z) = \frac{1}{z-1} , \; R_1(m,z) = \frac{e^{-amT}}{(z-e^{-aT})} ,$$

$$S_1(z) = \frac{z}{(z-1)(z-e^{-aT})} , \; D_1(z) = 0 \text{ and } J_1(z) = 1. \tag{3.3.55}$$

The associated transform $K_1(m,z)$ is now obtained. The second term in eqn.(3.3.52) is obtained, using complex translation theorem, as

$$= - e^{aT}\{B_1(m,ze^{aT}) - \lim_{z\to\infty} B_1(m,z)\}$$

$$= - \frac{e^{-2amT}}{(z - e^{-2aT})} \tag{3.3.56a}$$

The third term is obtained, using complex translation and real discrete convolution theorems, as

$$= -aT\, F_1(m,z)\Big[\{B_1(z) - \lim_{z\to\infty} B_1(z)\} + e^{aT}\{C_1(ze^{aT}) - \lim_{z\to\infty} C_1(z)\}\Big]$$

$$= \frac{-2aT\, e^{-amT}\{mz + e^{-aT}(1-m)\}}{(z - e^{-aT})^2} \tag{3.3.56b}$$

The fourth term is obtained, using the complex convolution and the real discrete convolution theorems, as

$$= -(1 - e^{-aT})\, R_1(m,z)\{D_1(z) + S_1(z) * J_1(z)\} = 0 \tag{3.3.56c}$$

since $D_1(z) = 0$ and $J_1(z) = 1$.

Then, the associated transform $Y_1(m,z)$ may be obtained from eqns. (3.3.52) and (3.3.54) to (3.3.56), as

$$Y_1(m,z) = z\,\frac{\Big[(z - e^{-aT})^2\{z(1 - e^{-2amT}) + (e^{-2amT} - e^{-2aT})\} - 2aT\, e^{-amT}\{mz + e^{-aT}(1-m)\}(z-1)(z - e^{-2aT})\Big]}{a^3(z-1)(z-e^{-2aT})(z - e^{-aT})^2} \tag{3.3.57}$$

which is the required output modified z transform.

It should be noted that a fairly lengthy procedure has been followed in order to demonstrate the use of most of the theorems developed here. However, the associated transform of $Y_3(m,z_1,z_2,z_3)$ can be obtained directly by using the theorems in two stages.

3.4 Illustration of the Use of the Transform methods

Finally, this section illustrates the use of the transform methods developed so far, by means of two nonlinear systems, in which one of the systems is followed by a smoothing network(linear system) and the other one is not followed by a smoothing network.

3.4.1 Analysis of a Nonlinear System with 3^{rd} Order Kernel followed by a Smoothing Network

Consider, first, the nonlinear system with a zero-order hold, shown in Fig.3.4, for which the output transform for a sampled step input is given by eqn.(3.3.57) and is repeated here for convvenience.

$$Y_1(m,z) = z\,\frac{\left[(z-e^{-aT})^2\{z(1-e^{-2amT})+(e^{-2amT}-e^{-2aT})\}-2aT\,e^{-amT}\{mz+e^{-aT}(1-m)\}(z-1)(z-e^{-2aT})\right]}{a^3(z-1)(z-e^{-2aT})(z-e^{-aT})^2}$$

$$0 \leqslant m < 1 \qquad (3.4.1)$$

From the linear system theory, the inverse of eqn.(3.4.1) gives the output between sampling instants, as

$$y_3(\langle i+m\rangle T) = \frac{1}{a^3}\{1-e^{-2a(i+m)T}-2aT(i+m)\,e^{-a(i+m)T}\}$$

$$0 \leqslant m < 1 \qquad (3.4.2)$$

The output at the sampling instants may, then, be obtained as

$$y_3(iT) = y_3(\langle i+m\rangle T)\Big|_{m=0} = \frac{1}{a^3}(1-e^{-2aiT}-2aiT\,e^{-aiT}) \qquad (3.4.3)$$

For a=1 and T=1 sec., the output at and between the sampling instants, for $0 \leqslant m < 1$, as obtained from eqns.(3.4.3) and (3.4.2), respectively, are shown in Fig.3.5. These results agree with the continuous-time output within reasonable limits.

3.4.2 Example 2 - Analysis of a Nonlinear System, with 3^{rd} Order Kernel, not followed by a Smoothing Network

Consider the system of Fig.3.6, in which the 3^{rd} order kernel is

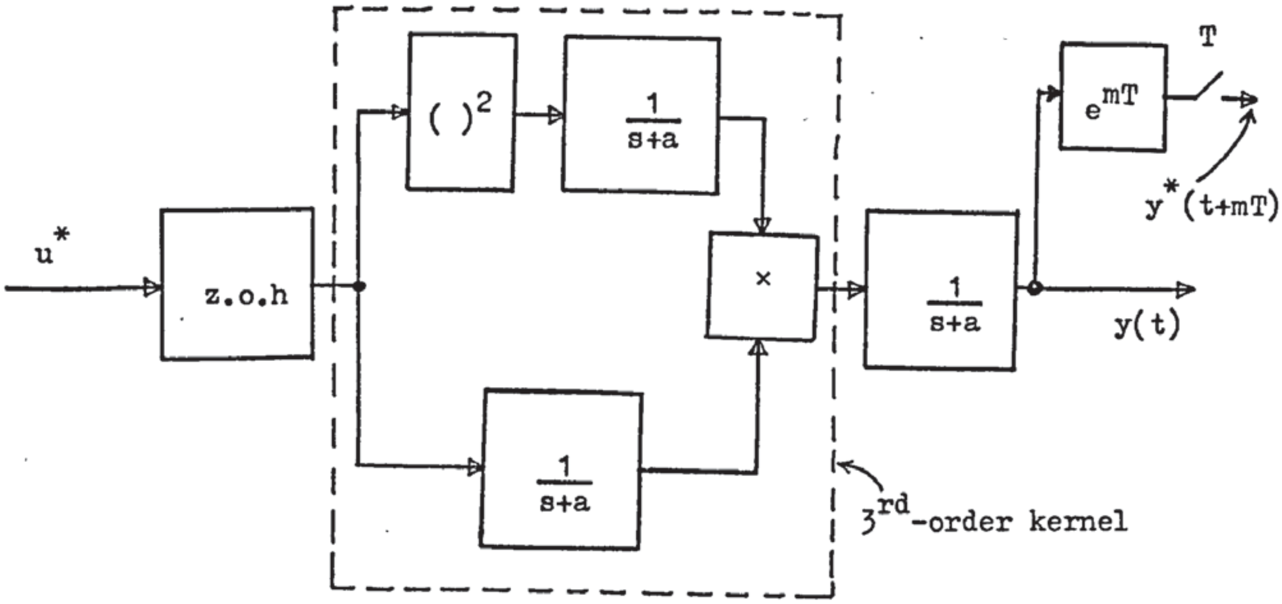


Fig.3.4 Nonlinear system with 3rd-order kernel preceded by zero-order hold and followed by a smoothing network.

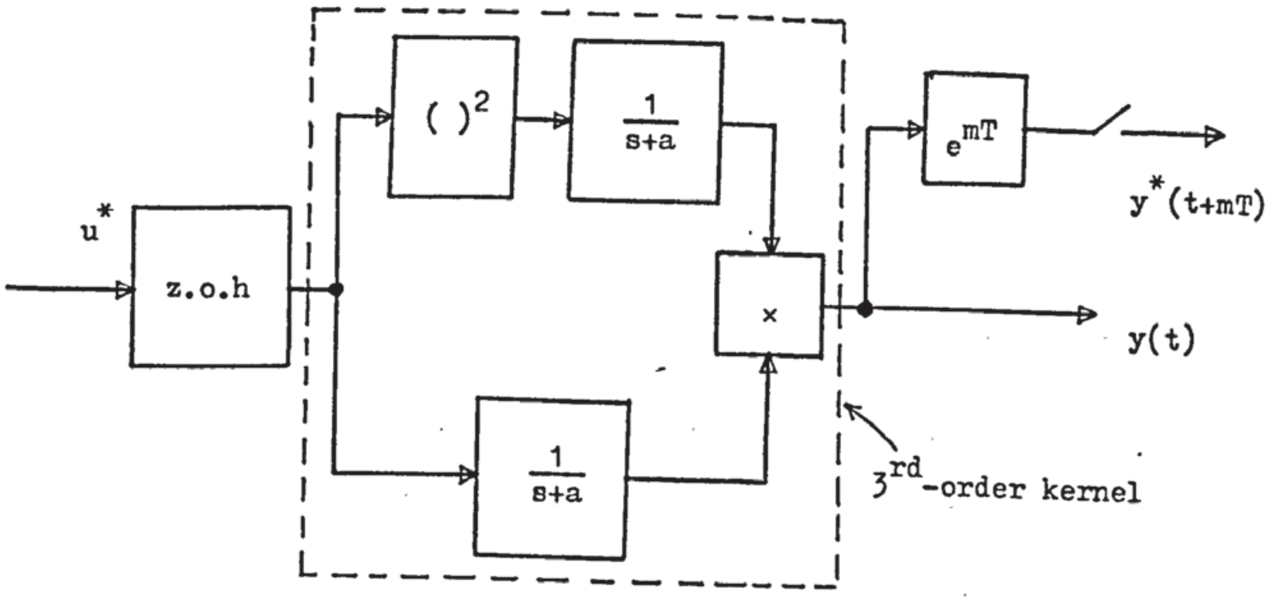


Fig.3.6 3rd-order kernel not followed by linear system.

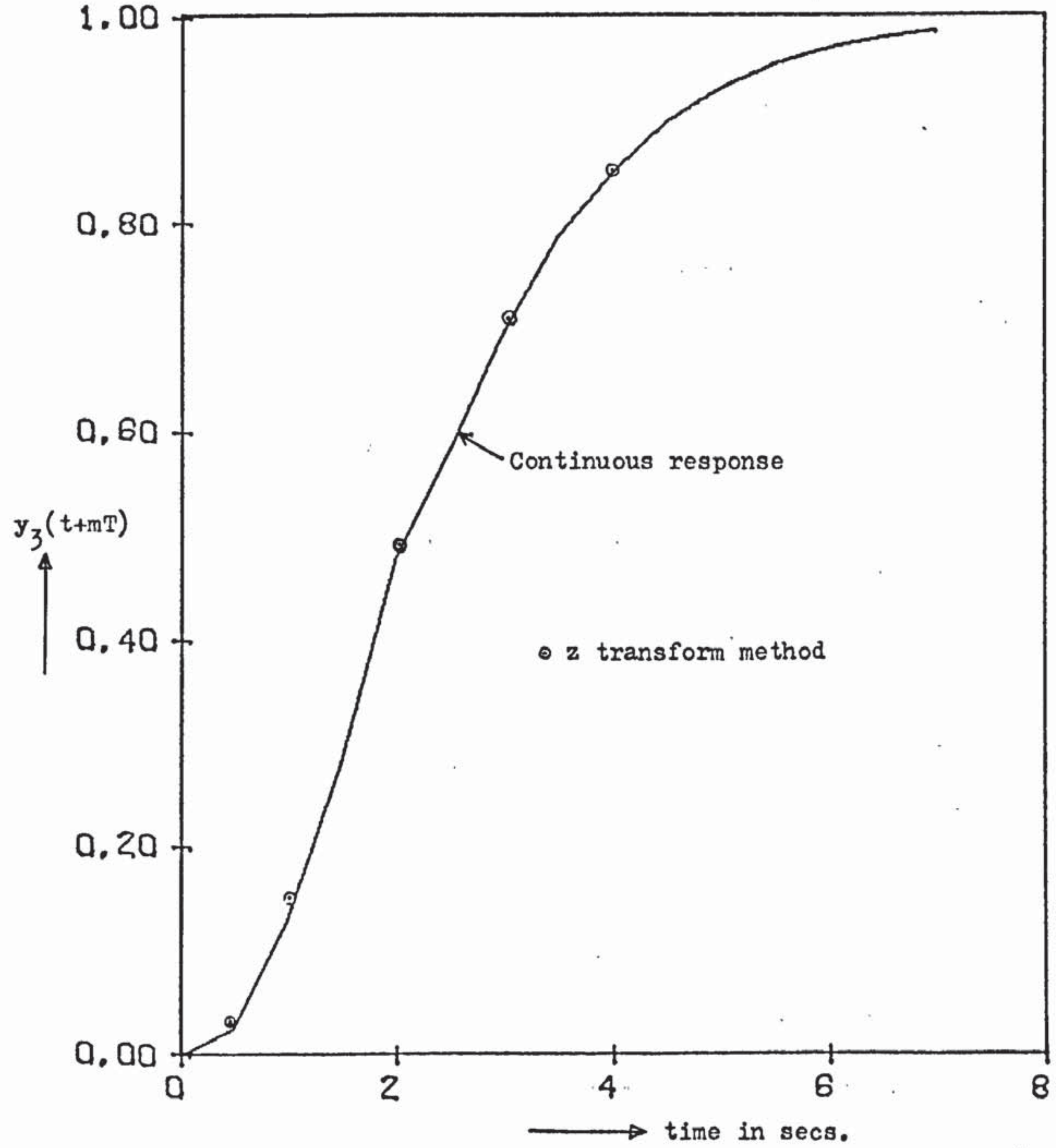


Fig.3.5 The step response of the system shown in Fig.3.4.

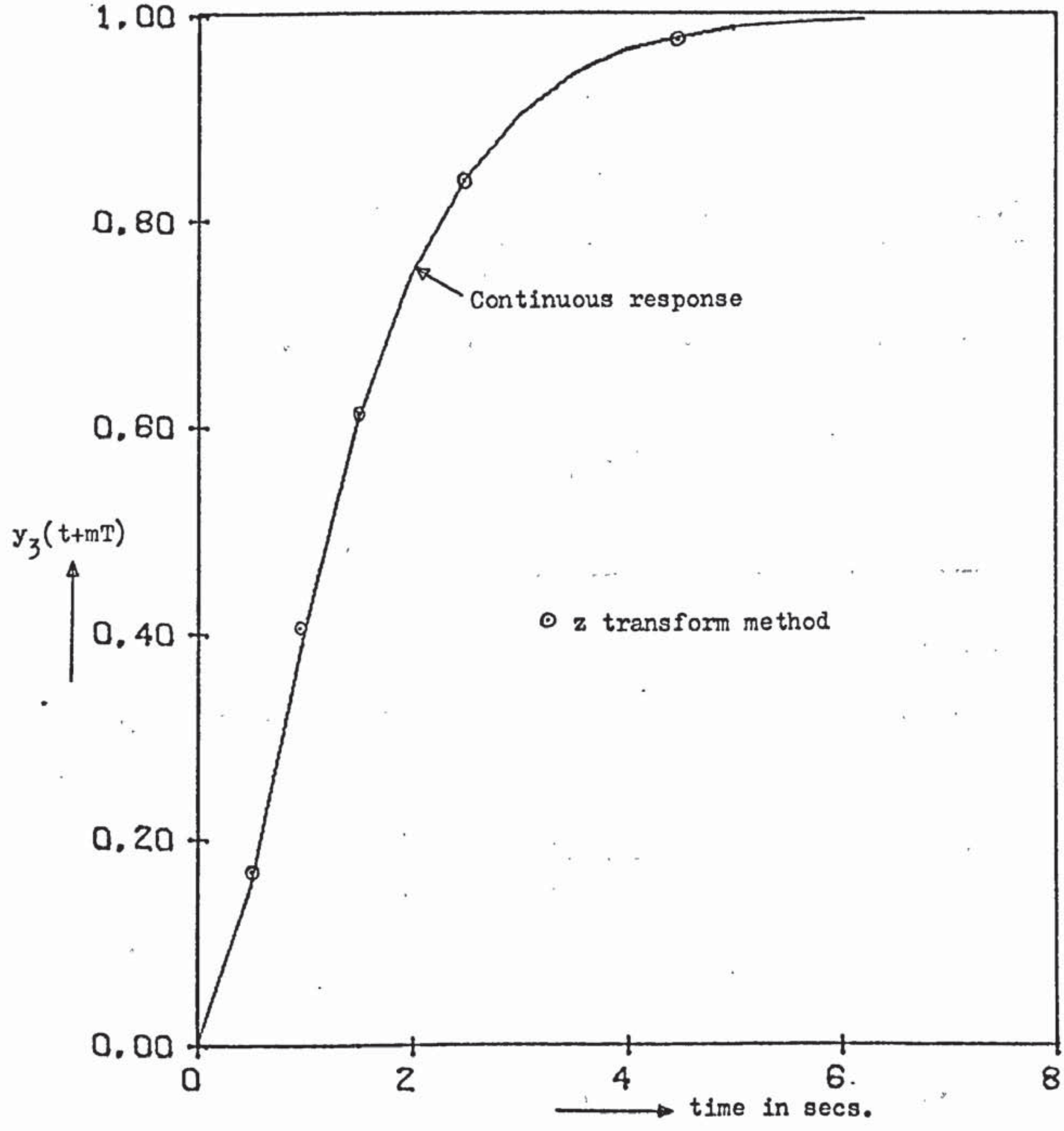


Fig.3.7 The step response of the system shown in Fig.3.6.

not followed by a linear system. For this system $H_3(s_1,s_2,s_3)$ is given by

$$H_3(s_1,s_2,s_3) = \frac{1}{s_1s_2s_3(s_1+a)(s_2+s_3+a)}$$

and $P_3(m,z_1,z_2,z_3)$ is then given by

$$P_3(m,z_1,z_2,z_3) = H_3(m,z_1,z_2,z_3)\prod_{r=1}^{3}\left(\frac{z_r-1}{z_r}\right)$$

$$= \frac{\{z_1(1-e^{-amT})+(e^{-amT}-e^{-aT})\}\{z_2z_3(1-e^{-amT})+(e^{-amT}-e^{-aT})\}}{a^2(z_1-e^{-aT})(z_2z_3-e^{-aT})}$$

If the input is a sampled step with transform $\frac{z}{z-1}$, then the output transform is given by

$$Y_3(m,z_1,z_2,z_3) = \frac{z_1z_2z_3\{z_1(1-e^{-amT})+(e^{-amT}-e^{-aT})\}\{z_2z_3(1-e^{-amT})+(e^{-amT}-e^{-aT})\}}{a^2(z_1-1)(z_1-e^{-aT})(z_2-1)(z_3-1)(z_2z_3-e^{-aT})} \tag{3.4.4}$$

The associated transform $Y_1(m,z)$ is obtained using the theorems of association-of-variables, as

$$Y_1(m,z) = \frac{z}{a^2}\left\{\frac{1}{(z-1)} - \frac{2e^{-amT}}{(z-e^{-aT})} + \frac{e^{-2amT}}{(z-e^{-2aT})}\right\} \tag{3.4.5}$$

The output between sampling instants is obtained, by taking inverse transform of $Y_1(m,z)$, as

$$y_3(\langle i+m\rangle T) = \frac{1}{a^2}\{1-e^{-a(i+m)T}\}^2 \quad , \quad 0 \leq m < 1 \tag{3.4.6}$$

and the output at sampling instants is then given by

$$y_3(iT) = y_3(\langle i+m\rangle T)\Big|_{m=0} = \left\{\frac{1-e^{-aiT}}{a}\right\}^2 \tag{3.4.7}$$

For the same values of a and T, as given before, the continuous output and the output at the discrete instants of time are calculated using eqns. (3.4.10) and (3.4.11), respectively, and shown in Fig. 3.7.

It may be seen from this figure that the results given by the z transform and modified z transform methods, for $0 \leq m < 1$, agree well with the continuous-time output of the system.

3.5 Conclusions

The multidimensional modified z transform of a Volterra kernel or a kernel cascaded with a data-hold device, can be obtained by applying the sequential process developed here, to the multidimensional Laplace transform of the kernel or the cascade combination of the kernel and the data-hold device, respectively, which is easily synthesised for a large class of nonlinear systems. The procedure at each stage of the process is a simple one of calculating residues. For systems separated by a sampler, the multidimensional modified z transform of the cascade is given by the product of the multi-dimensional modified z transform of the second system and the multi-dimensional z transform of the first system. The reason for this is that the output of the second system is modified whereas the output of the first system is not modified.

The intersampled response $y(\langle i+m \rangle T)$ of a nonlinear system with a given sampled-data input, can be obtained, for $0 \leq m < 1$, by using the theorems of association-of-variables, developed here; one or more theorems must be used for the association-of-variables, which, in many cases, may be carried out by inspection. The output at the sampling instants $y(iT)$, can also be obtained from $y(\langle i+m \rangle T)$ by letting $m=0$. Thus, the theorems are as good as the sequential process, if not better, in obtaining the associated transform. However, the theorems may also be used for other applications, as shown in chapter 7.

The use of the transform methods is demonstrated by several examples, in which it is observed that the intersampled response given by the modified z transform method is same as the continuous-time output of the system. A fairly comprehensive table of multidimensional z and modified z transforms and their associated transforms is given in Appendix A.3.2. This table is quite useful because the multidimensional z and modified z transforms contained in the table are derived for Volterra kernels which occur frequently in the analysis of nonlinear sampled-data systems represented by block-diagrams. The table also contains associated transforms which represent the output transforms of some nonlinear systems having impulse, step and exponential forcing functions.

CHAPTER 4

BLOCK DIAGRAM ALGEBRA FOR NONLINEAR DISCRETE SYSTEMS

4.1 Introduction

Volterra functional representation, not only gives an explicit input-output relation for a nonlinear system, but also facilitates the combination of systems. George[13] has presented an operator algebra for combination of continuous systems and others who have contributed to algebra of continuous nonlinear systems include Brilliant[12], Zames[14], Bansal[17] and Barker[87]. The aim of this chapter is to extend and generalise their results to nonlinear discrete systems, in which the zero-order subsystems are assumed to be present. A set of six basic operations are described by which the generalised Volterra functional expansion of various nonlinear sampled-data system connections, in which the inputs to subsystems are applied through data-hold devices, may be obtained. These operations may also be used for obtaining the explicit input-output kernels of a nonlinear feedback system in which the subsystems may or may not be separated by a sampler.

The analysis of nonlinear feedback systems with various locations of samplers, is given in Appendix A.4.

4.2 Generalised Volterra Series Representation of Signals

A block-diagram algebra for the combination of nonlinear discrete systems may be developed if it is assumed that each signal in the system, at the sampling instants, has a generalised Volterra series expansion as follows:

$y = y_0$	- a zero-order component which is constant for all time
$+\ y_1(kT)$	- a first-order component which represents the linear behaviour of the system at the sampling instants
$+\ y_2(k_1T, k_2T)$	- a second-order component which represents the quadratic behaviour of the system at the sampling instants when $k_1 = k_2 = k$

$+ y_3(k_1T, k_2T, k_3T)$ - a third-order component which represents the cubic behaviour of the system at the sampling instants when $k_1 = k_2 = k_3 = k$

+ etc. (4.2.1)

In z transforms, this may be represented as

$$Y = Y_0 + Y_1(z) + Y_2(z_1, z_2) + Y_3(z_1, z_2, z_3) + \text{etc.} \tag{4.2.2}$$

where it is to be noted, in particular, that $Y_0 = y_0$.

In the next section, the input-output relationships for six basic operations are developed. Though the operations described here are primarily valid for single-input, single-output systems, they may be easily extended to multivariable systems as well. The operations are also extended for various system combinations, in order to obtain their generalised Volterra kernels. All operations are described in transform domain, for convenience.

The following assumptions are made in this development.

(a) All kernels are symmetric,

(b) All samplers are ideal and have the same rate, and

(c) All the signals and systems have a generalised Volterra series expansions.

If an unsymmetrical kernel is present, then it may be symmetrised[13] by replacing it by the arithmetic mean of the kernels obtained by all permutations of its arguments. For example, $J_2(z_1, z_2)$ may be symmetrised by forming

$$J_2(z_1, z_2) = \frac{1}{2}\left[J_2(z_1, z_2) + J_2(z_2, z_1)\right] \tag{4.2.3}$$

and $J_3(z_1, z_2, z_3)$ may be symmetrised by forming

$$J_3(z_1, z_2, z_3) = \frac{1}{6}\left[J_3(z_1, z_2, z_3) + J_3(z_1, z_3, z_2) + J_3(z_2, z_1, z_3) + J_3(z_2, z_3, z_1) + J_3(z_3, z_1, z_2) + J_3(z_3, z_2, z_1)\right] \tag{4.2.4}$$

The algebraic rules, which these operations obey, are similar to those developed by George[13] for continuous systems, and are indicated at the end of each operation.

4.3 Block Diagram Algebra

In this section, the following six operations are described to facilitate various combinations of nonlinear discrete systems :

(a) Summing operation
(b) Multiplication operation
(c) Linear system operation
(d) Nonlinear system operation
(e) Cascade operation
(f) Feedback system operation.

4.3.1 Summing Operation

The summing operation of two signals is illustrated in Fig.4.1(a), in which r^* and s^* are input signals to the adder whose output is y^*. In operator notation, this may be represented as

$$y^* = r^* + s^* \tag{4.3.1}$$

If r^*, s^* and y^* are in the form of generalised Volterra series expansions, as eqn.(4.2.1), then substituting these expansions in eqn.(4.3.1) and equating terms of equal order and transforming, yields

$$\begin{aligned}
Y_0 &= R_0 + S_0 \\
Y_1(z) &= R_1(z) + S_1(z) \\
Y_2(z_1,z_2) &= R_2(z_1,z_2) + S_2(z_1,z_2) \\
Y_3(z_1,z_2,z_3) &= R_3(z_1,z_2,z_3) + S_3(z_1,z_2,z_3) \\
&\text{etc.}
\end{aligned} \tag{4.3.2}$$

In terms of the system combination, this operation means that r^* and s^* are the sampled outputs of two subsystems J and K, for the same input u^*, and the outputs are added to give y^* as shown in Fig.4.1(b). Then, y^* may be regarded as the sampled output of a system M with input u^*. Then, the operation may be written algebraically as[88]

$$M^* = J^* + K^* \tag{4.3.3}$$

The combined system M^* has the degree of the highest degree subsystem J^* or K^*. The commutative law ($J^*+K^* = K^*+J^*$) and the associative law

$\boxed{(J^*+K^*)+N^* = J^*+(K^*+N^*)}$ hold for summing operation, as in continuous systems .

4.3.2 Multiplication Operation

If r^* and s^* are two inputs to a multiplier, then its output y^* may be obtained from the operation

$$y^* = r^* . s^* \tag{4.3.4}$$

This operation is shown in Fig.4.2(a). The terms of equal order, of the above equation in transforms are given by

$$\begin{aligned}
Y_0 &= R_0S_0 \\
Y_1(z) &= R_0S_1(z) + R_1(z)S_0 \\
Y_2(z_1,z_2) &= R_0S_2(z_1,z_2) + R_1(z_1)S_1(z_2) + R_2(z_1,z_2)S_0 \\
Y_3(z_1,z_2,z_3) &= R_0S_3(z_1,z_2,z_3) + R_1(z_1)S_2(z_2,z_3) + R_2(z_1,z_2)S_1(z_3) \\
&\quad + R_3(z_1,z_2,z_3)S_0
\end{aligned} \tag{4.3.5}$$

etc.

This result may be easily proved by identifying the components of each order in $r^*. s^*$. For example, the second-order term in the product $r^*. s^*$ is

$$y_2(k_1T,k_2T) = r_0s_2(k_1T,k_2T)+ r_1(k_1T)s_1(k_2T)+ r_2(k_1T,k_2T)s_0$$

Taking the z transform of the above equation, yields

$$Y_2(z_1,z_2) = R_0S_2(z_1,z_2) + R_1(z_1)S_1(z_2) + R_2(z_1,z_2)S_0$$

The other terms of eqn.(4.3.5) may be similarly obtained. However, in terms of the system combination, this operation involves putting the same input u^* into J and K, and multiplying the two sampled outputs r^* and s^* in a multiplier, to yield y^* as shown in Fig.4.2(b). Then, one may write the operation algebraically as[88]

$$M^* = J^*. K^* \tag{4.3.6}$$

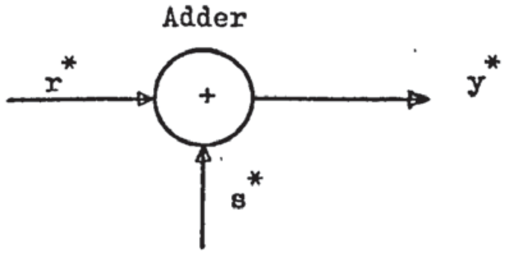


Fig.4.1(a) Summing operation of two signals.

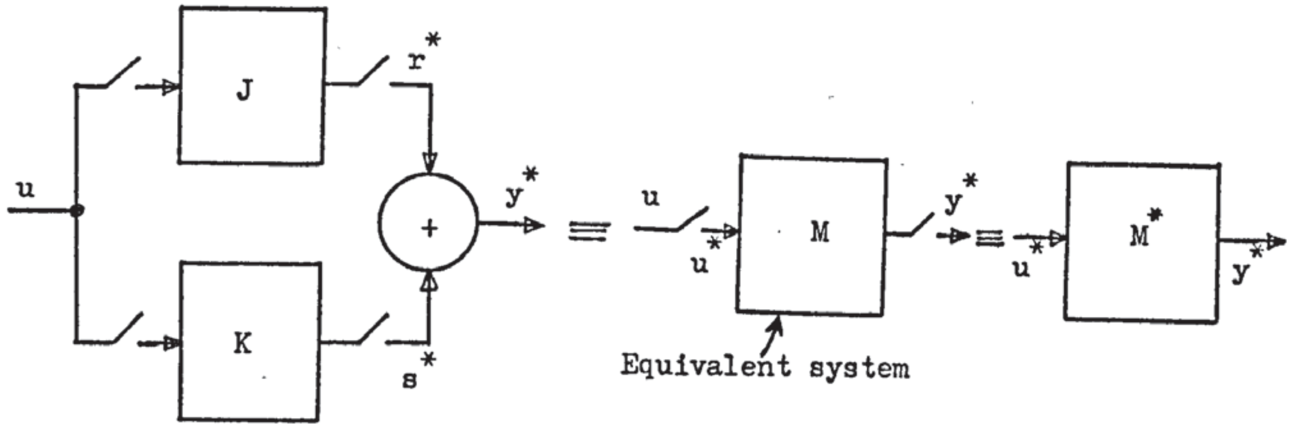


Fig.4.1(b) Summation of two discrete systems.

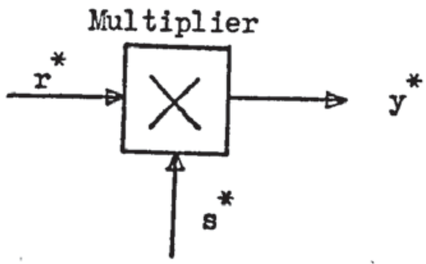


Fig.4.2(a) Multiplication of two signals.

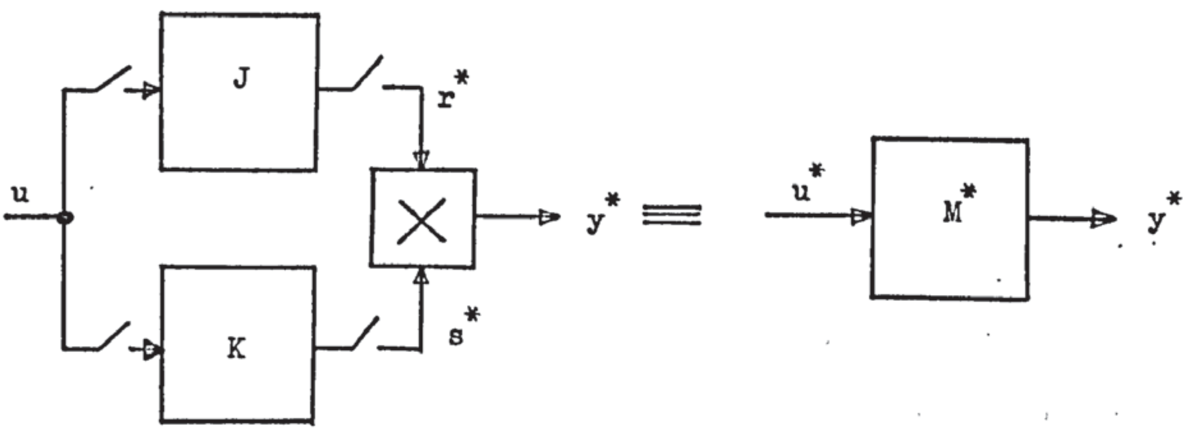


Fig.4.2(b) Multiplication of two digital systems.

where the input and the output of M^* are u^* and y^*, respectively. The components of the above equation may be obtained as before. If J^* and K^* are systems of order p and q, respectively, then the order of M^* is (p+q).

The commutative law ($J^*. K^* = K^*. J^*$), the distributive law $\{J^*.(K+N)^* = J^*. K^* + J^*. N^*\}$ and the associative law $\{(J^*.K^*). N^* = J^*.(K^*.N^*)\}$ hold for the multiplication operation, as in continuous systems . The multiplication of two linear discrete systems produces the simplest nonlinear discrete system and they are used later in the digital simulation of nonlinear systems. The multiplication operation also illustrates the advantage of system representation in the multidimensional transform domain.

4.3.3 Linear System Operation

This operation is illustrated in Fig.4.3(a), in which r^* is a sampled input to a linear system W_1 whose sampled output is y^* and represents a linear system operating on a signal r^* to yield y^*. In operator notation, this is written as

$$y^* = (w_1 \otimes r^*)^* = w_1^* \otimes r^* \tag{4.3.7}$$

The components of the above equation, in z transforms, are

$$\begin{aligned}
&Y_0 = W_1(1)R_0\\
&Y_1(z) = W_1(z)R_1(z)\\
&Y_1(z_1,z_2) = W_1(z_1z_2)R_2(z_1,z_2)\\
&Y_3(z_1,z_2,z_3) = W_1(z_1z_2z_3)R_3(z_1,z_2,z_3)\\
&\qquad\text{etc.}
\end{aligned} \tag{4.3.8}$$

This is easily proved by considering components of each order in the real-time relationship of eqn.(4.3.7), and transforming. For example, the second order term of $w_1^* \otimes r^*$ is

$$y_2(k_1T,k_2T) = \sum_{n=0}^{\infty} w_1(nT)r_2(\langle k_1-n\rangle T,\langle k_2-n\rangle T)$$

z transforming the above equation gives

$$Y_2(z_1,z_2) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\sum_{n=0}^{\infty} w_1(nT)r_2(<k_1-n>T,<k_2-n>T)z_1^{-k_1}z_2^{-k_2}$$

$$= \sum_{n=0}^{\infty} w_1(nT)(z_1z_2)^{-n}\sum_{k_1=n}^{\infty}\sum_{k_2=n}^{\infty} r_2(<k_1-n>T,<k_2-n>T) \times z_1^{-(k_1-n)} z_2^{-(k_2-n)}$$

$$= W_1(z_1z_2)\, R_2(z_1,z_2) \qquad (4.3.9)$$

The other terms of (4.3.8) may be similarly obtained.

In terms of system combination, the signal r^* may be regarded as the signal generated by a nonlinear system J, as shown in Fig.4.3(b). Then, this operation describes a system M^* with input u^* and output y^*, where M^* is given, in operator form, by

$$M^* = W_1^* \otimes J^* \qquad (4.3.10)$$

This may be easily obtained as shown below. The sampled output y^* of W_1 is given by $y^* = W_1^* \otimes r^* = W_1^* \otimes J^*(u^*)$. But, $y^* = M^*(u^*)$. Comparison of these two equations gives eqn.(4.3.10). The components of eqn. (4.3.10) are given, in modified z transform, by

$$\begin{aligned}
M_0 &= W_1(m,1)\, J_0 \\
M_1(m,z) &= W_1(m,z)\, J_1(z) \\
M_2(m,z_1,z_2) &= W_1(m,z_1z_2)\, J_2(z_1,z_2) \\
M_3(m,z_1,z_2,z_3) &= W_1(m,z_1z_2z_3)\, J_3(z_1,z_2,z_3) \quad ,
\end{aligned} \qquad (4.3.11)$$

$$\text{etc.,} \qquad 0 \leqslant m < 1$$

The above equations are valid for the case of asynchronous input-output sampling only. For synchronous sampling case, m must be equated to zero in the above equations. It may be noted that the variable m is found only in W_1^* and not in J^*. This is because, the output of J^* is not modified and the input to $w_1(\tau)$ is received only at the sampling instants. If J^* is a system of order p, then the order of system M^* is also p, since W_1^* is a linear system.

The distributive law $\{W_1^* \otimes (J^* + K^*) = (W_1^* \otimes J^* + W_1^* \otimes K^*)\}$ holds for linear system operation, and it may be easily shown that, in general,

$W_1^* \otimes J^* \neq J^* \otimes W_1^*$

However, if J^* is a linear system, then the commutative law $\{W_1^* \otimes J_1^* = J_1^* \otimes W_1^*\}$ also holds for linear system operation.

4.3.4 Nonlinear System Operation

This operation represents a nonlinear system operating on a signal r^* as shown in Fig.4.4(a), in which y^* is the result of the operation. The operation is demonstrated here for second order nonlinear system and may be easily extended to an n^{th} order case. Then, the operation may be symbolically written as

$$y^* = (w_2 \circledast r^*)^* = w_2^* \circledast r^* \tag{4.3.12}$$

The components, of different order, of eqn.(4.3.12) in z transforms, are identified to be

$$\begin{aligned}
Y_0 &= W_2(1,1) \prod_{k=1}^{2} R_0 \\
Y_1(z) &= 2R_0\ W_2(1,z)\ R_1(z) \\
Y_2(z_1,z_2) &= 2R_0W_2(1,z_1z_2)R_2(z_1,z_2) + W_2(z_1,z_2) \prod_{k=1}^{2} R_1(z_k) \qquad (4.3.13) \\
Y_3(z_1,z_2,z_3) &= 2R_0W_2(1,z_1z_2z_3)R_3(z_1,z_2,z_3) + 2W_2(z_1,z_2z_3)R_1(z_1)R_2(z_2,z_3)
\end{aligned}$$

etc.

Identifying components of each order in the real-time relationship $w_2^* \circledast r^*$ and z transforming, gives eqn.(4.3.13). For example, the second order term in $w_2^* \circledast r^*$ is given by

$$y_2(k_1T,k_2T) = 2r_0 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} w_2(n_1T,n_2T) r_2(\langle k_1-n_2\rangle\ T, \langle k_2-n_2\rangle\ T)$$

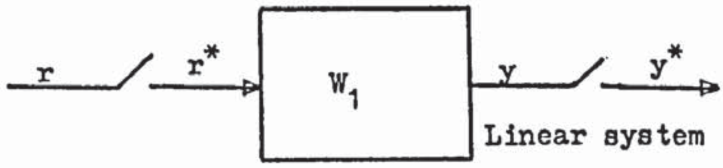


Fig.4.3(a) A schematic diagram illustrating the Linear System Operation.

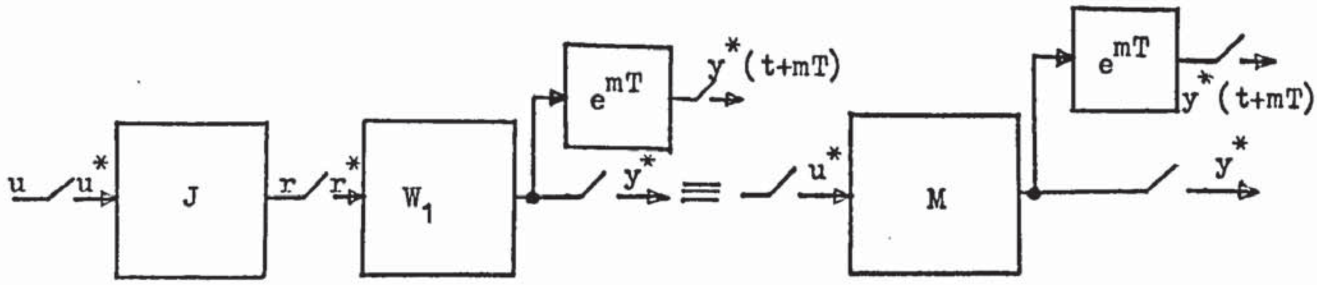


Fig.4.3(b) Cascade connection of systems J and W_1, separated by a sampler

Fig.4.4(a) A schematic diagram illustrating nonlinear system operation.

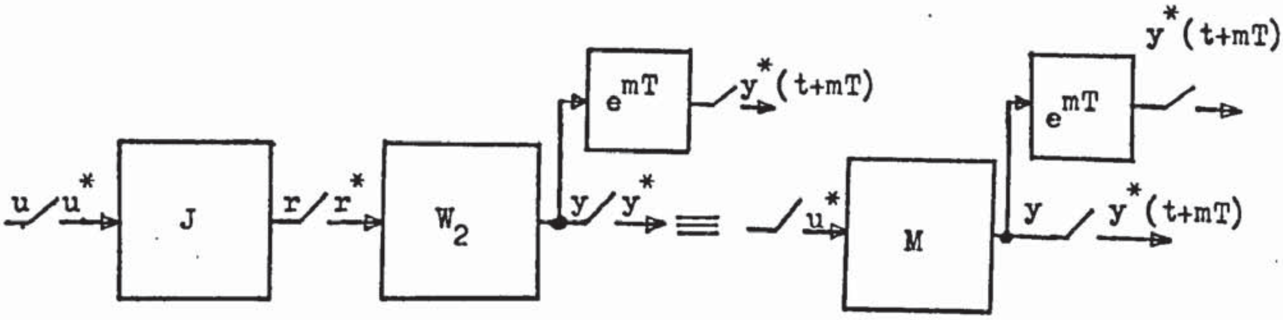


Fig.4.4(b) Cascade connection of two nonlinear systems, separated by a sampler.

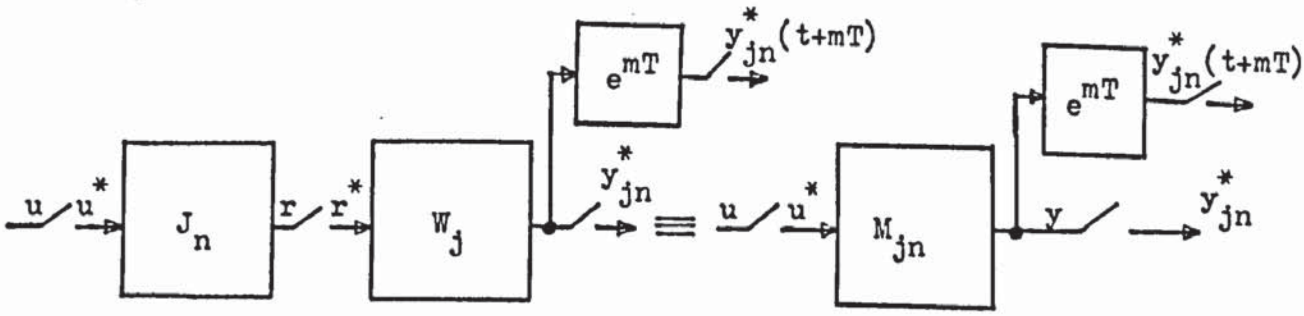


Fig.4.4(c) Cascading of a n^{th} order system J_n and a j^{th} order system W_j with a sampler between them.

$$+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} w_2(n_1T, n_2T) r_1(<k_1-n_1> T) r_1(<k_2-n_2> T)$$

z transforming the above equation gives

$$Y_2(z_1, z_2) = 2R_0 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} w_2(n_1T, n_2T) r_2(<k_1-n_2> T, <k_2-n_2> T) \times z_1^{-k_1} z_2^{-k_2}$$

$$+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} w_2(n_1T, n_2T) r_1(<k_1-n_1> T) r_2(<k_2-n_2> T) \times z_1^{-k_1} z_2^{-k_2}$$

$$= 2R_0 \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} w_2(n_1T, n_2T)(z_1 z_2)^{-n_2}$$

$$\times \sum_{k_1=n_2}^{\infty} \sum_{k_2=n_2}^{\infty} r_2(<k_1-n_2>T, <k_2-n_2>T)\, z_1^{-(k_1-n_2)} z_2^{-(k_2-n_2)}$$

$$+ \sum_{n_1=0}^{\infty} \sum_{n_2=0}^{\infty} w_2(n_1T, n_2T)\, z_1^{-n_1} z_2^{-n_2}$$

$$\times \sum_{k_1=n_1}^{\infty} \sum_{k_2=n_2}^{\infty} r_1(<k_1-n_1>T) r_1(<k_2-n_2>T)\, z_1^{-(k_1-n_1)} z_2^{-(k_2-n_2)}$$

$$= 2R_0 W_2(1, z_1 z_2) R_2(z_1, z_2) + W_2(z_1, z_2) R_1(z_1) R_1(z_2)$$

The other terms of eqn.(4.3.13) may be similarly obtained.

If, however, r^* is the sampled output of a nonlinear system J, for a sampled input u^*, then this operation represents a cascade connection of J and W_2, separated by a sampler, as shown in Fig.4.4(b). Then, y^* may be treated as the sampled output of an equivalent system M^*, with input u^*, where M^* is given by

$$M^* = W_2^* \circledast J^* \qquad (4.3.14)$$

The components of the generalised Volterra series expansion of M^*, in modified z transform, are then given by

$$M_0 = W_2(m, 1, 1) \prod_{k=1}^{2} J_0$$

$$M_1(m,z) = 2J_0W_2(m,1,z)J_1(z)$$

$$M_2(m,z_1,z_2) = 2J_0W_2(m,1,z_1z_2)J_2(z_1,z_2) + W_2(m,z_1,z_2)\prod_{k=1}^{2} J_1(z_k)$$

$$M_3(m,z_1,z_2,z_3) = 2J_0W_2(m,1,z_1z_2z_3)J_3(z_1,z_2,z_3) + 2W_2(m,z_1,z_2z_3)J_1(z_1)J_2(z_2,z_3) \tag{4.3.15}$$

etc., $0 \leqslant m < 1$

In general, when a n^{th} order subsystem J_n and a j^{th} order subsystem W_j are cascaded with a sampler between them as shown in Fig.4.4(c), then the M.D.M.Z.T of the resulting system M, which is of order jn is given by

$$M_{jn}(m,z_1,z_2,\ldots,z_n,z_{n+1},\ldots,z_{jn})$$

$$= W_j(m,\prod_{k=1}^{n} z_k,\ldots,\prod_{k=(jn-n+1)}^{jn} z_k)J_n(z_1,\ldots,z_n)J_n(z_{n+1},\ldots,z_{2n}) \times \ldots J_n(z_{jn-n+1},\ldots,z_{jn})$$

$$0 \leqslant m < 1 \tag{4.3.16}$$

The reason for m appearing in W_j only, is obvious. It should be noted that , when j=1 in the above equation, the nonlinear system operation reduces to the linear system operation.

The associative law, $J^* \circledast (K^* \circledast N^*) = (J^* \circledast K^*) \circledast N^*$, holds for non-linear system operation, whereas the commutative law is not valid for this operation, i.e; $J^* \circledast K^* \neq K^* \circledast J^*$, in general. However, the distributive law, though not valid, in general, in cases like $N^* \circledast (J^* + K^*) \neq N^* \circledast J^* + N^* \circledast K^*$ and $N^* \circledast (K^*.J^*) \neq (N^* \circledast K^*).(N^* \circledast J^*)$, is valid in the following cases.

$(N^* + K^*) \circledast J^* = N^* \circledast J^* + K^* \circledast J^*$ and

$(N^*.K^*) \circledast J^* = (N^* \circledast J^*).(K^* \circledast J^*)$

When three subsystems J^*, K^* and W^* are to be cascaded, then the nonlinear system operation has to be used twice successively to obtain the equivalent system. First, J^* is cascaded with K^* to yield N^* as $N^* = K^* \circledast J^*$. Then, N^* is cascaded with W^* which gives M^* as $M^* = W^* \circledast N^*$.

Thus, the rule to cascade n subsystems, is to use the nonlinear system operation (n-1) times successively to obtain the equivalent system.

4.3.5 Cascade Operation

The linear and nonlinear system operations developed in previous sub-sections represent cascade connection of two systems separated by a sampler. The cascade operation in this sub-section, means the cascade connection of two nonlinear systems without a sampler between them. This situation occurs quite frequently in sampled-data systems in which two continuous subsystems J and K may be cascaded without a sampler between them. In such a case, linear and nonlinear system operations may be used, only if the second subsystem is memoryless. However, if J and K are continuous nonlinear systems, then they may be cascaded by finding the M.D.L.T of their cascade combination, using the cascade relations available for continuous systems. Then, the corresponding M.D.M.Z.T of the cascade combination may be found by applying the sequential process, developed in Chapter 3, to the derived M.D.L.T.

For example, if J is a n^{th} order subsystem and K is a j^{th} order subsystem, as shown in Fig.4.5(a), then the M.D.L.T of their cascade combination is

$$M_{jn}(s_1,s_2,\ldots,s_n,s_{n+1},\ldots,s_{jn})$$
$$= K_j\Big(\sum_{k=1}^{n}s_k,\ldots,\sum_{k=jn-n+1}^{jn}s_k\Big)J_n(s_1,s_2,\ldots,s_n)J_n(s_{n+1},\ldots,s_{2n})$$
$$\times\ldots\ldots J_n(s_{jn-n+1},\ldots,s_{jn}) \tag{4.3.17}$$

and is shown in Fig.4.5(b). Then, the M.D.M.Z.T of the cascade combination $M_{jn}(m,z_1,z_2,\ldots,z_{jn})$ may be obtained by applying the sequential process to $M_{jn}(s_1,s_2,\ldots,s_{jn})$ and is shown in Fig.4.5(c). The validity of associative, commutative and distributive laws, for this case, is same as that for linear and nonlinear system operations. However, in addition to these laws, it is to be noted that

$$\mathcal{Z}\left[K_j(s)\circledast J_n(s)\right] \neq K_j(z)\circledast J_n(z)$$

This operation may be extended to cascade more than two subsystems without samplers between them. For example, to cascade the subsystems J, K and N without samplers between them, the cascade relation of continuous systems is used to obtain the equivalent system M as $M = N\circledast K\circledast J$. Then, M^* is obtained from M, using the sequential process.

4.3.6 Feedback System Operation

This operation represents a nonlinear feedback system operating on a signal u^*, the result of the operation being the signal y^* as shown in Fig.4.6(a), in which J and K are nonlinear subsystems. The equivalent open-loop system representing the given feedback system is shown in Fig. 4.6(b). Then, this operation may be regarded as a nonlinear system operation, in which L^* operates on u^* to yield y^*, provided that the explicit input-output kernels of L^* are obtained in terms of J^* and K^*. In obtaining the kernels of L^*, the operations described so far are used. The zero-order component in the signals is assumed to be equal to zero, and the zero-order subsystems are assumed to be absent, for simplicity.

Since J^* and K^* are nonlinear subsystems, the explicit input-output relationship of the feedback system results in an infinite number of Volterra kernels. However, the feedback system may be characterised by a finite number of kernels if the nonlinearities are not too violent. Let the output of the feedback system shown in Fig.4.6(a) be characterised by the first three terms of its Volterra series expansion as

$$Y = Y_1(z)+ Y_2(z_1,z_2)+ Y_3(z_1,z_2,z_3) \tag{4.3.18}$$

The feedback signal r^* may be obtained from the nonlinear system operation,

$$r^* = K^*\circledast y^* \tag{4.3.19}$$

which gives the components of R, in transforms, as

$$R_1(z) = K_1(z)Y_1(z)$$

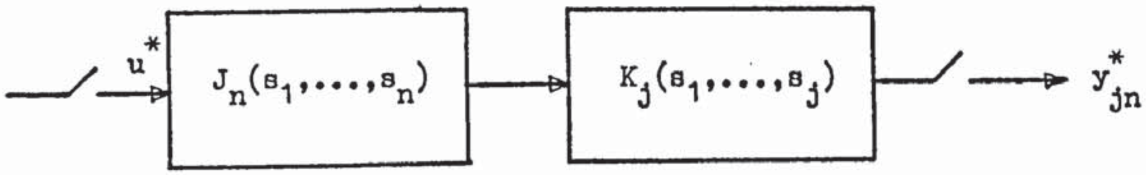


Fig.4.5(a) Cascade connection of two continuous systems without a sampler between them.

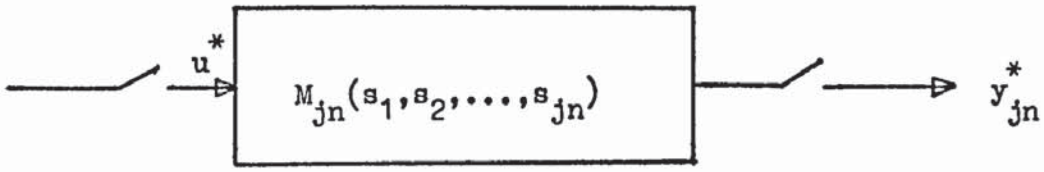


Fig.4.5(b) Equivalent system that represents the cascade combination of Fig.4.5(a).

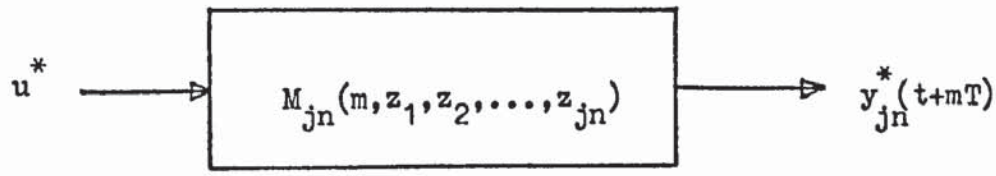


Fig.4.5(c) Representation of the cascade combination in modified z transform domain.

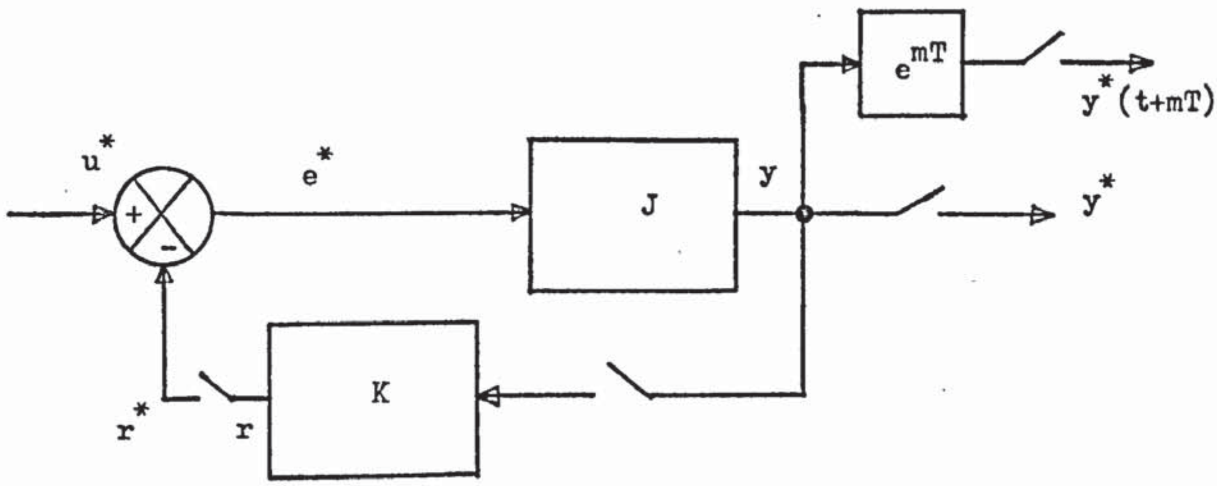


Fig.4.6(a) A typical feedback nonlinear discrete system.

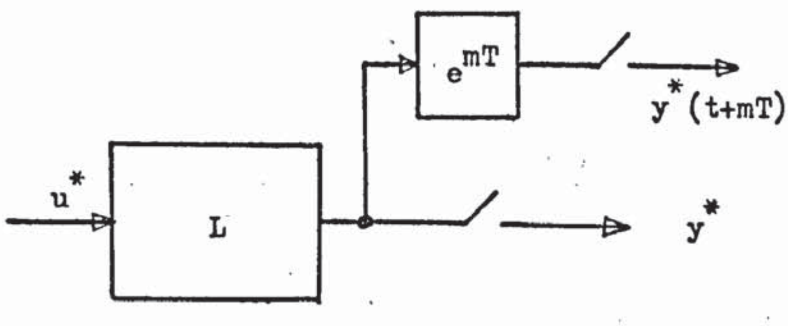


Fig.4.6(b) Equivalent open-loop system representing feedback system of Fig.4.6(a).

$$R_2(z_1,z_2) = K_1(z_1z_2)Y_2(z_1,z_2)+ K_2(z_1,z_2)Y_1(z_1)Y_1(z_2) \tag{4.3.20}$$
$$R_3(z_1,z_2,z_3) = K_1(z_1z_2z_3)Y_3(z_1,z_2,z_3)+ 2K_2(z_1,z_2z_3)Y_1(z_1)Y_2(z_2,z_3) + K_3(z_1,z_2,z_3)\prod_{p=1}^{3} Y_1(z_p)$$

The error signal e^* is given by the summing operation,

$$e^* = u^* - r^* \tag{4.3.21}$$

whose components, in z transforms, are given by

$$\begin{aligned} E_1(z) &= U_1(z)- R_1(z) \\ E_2(z_1,z_2) &= - R_2(z_1,z_2) \\ E_3(z_1,z_2,z_3) &= - R_3(z_1,z_2,z_3) \end{aligned} \tag{4.3.22}$$

The nonlinear system operation

$$y^* = J^* \circledast e^* \tag{4.3.23}$$

gives the sampled output y^*. Substituting eqns.(4.3.20) into eqns. (4.3.22) for components of R and the resulting equations into the components of eqn.(4.3.23) for components of E and then solving for the components of Y yield

$$Y_1(m,z) = \frac{J_1(m,z)\ U_1(z)}{\{1+ K_1(z)J_1(z)\}} \tag{4.3.24}$$

$$Y_2(m,z_1,z_2) = \frac{\{J_2(m,z_1,z_2)E_1(z_1)E_1(z_2)- J_1(m,z_1z_2)K_2(z_1,z_2)Y_1(z_1)Y_1(z_2)\}}{\{1+ K_1(z_1z_2)J_1(z_1z_2)\}} \tag{4.3.25}$$

and

$$Y_3(m,z_1,z_2,z_3) = \frac{\Big[J_3(m,z_1,z_2,z_3)\prod_{p=1}^{3} E_1(z_p)-2J_2(m,z_1,z_2z_3)E_1(z_1)\{K_1(z_2z_3).Y_2(z_2,z_3) +K_2(z_2,z_3)Y_1(z_2)Y_1(z_3)\} -J_1(m,z_1z_2z_3)\{K_3(z_1,z_2,z_3)\prod_{p=1}^{3}Y_1(z_p) + 2K_2(z_1,z_2z_3)Y_1(z_1)Y_2(z_2,z_3)\}\Big]}{\{1 + K_1(z_1z_2z_3)\ J_1(z_1z_2z_3)\}} \tag{4.3.26}$$

$E_1(z) = \dfrac{U_1(z)}{\{1 + K_1(z)J_1(z)\}}$, and $Y_1(m,z)$ and $Y_2(m,z_1,z_2)$ are given by eqns. (4.3.24) and (4.3.25), respectively. If L^* is the equivalent open-loop nonlinear system representing the feedback system shown in Fig.4.6(a), then the sampled output y^* is given by the nonlinear system operation

$$y^* = L^*(u^*) \tag{4.3.27}$$

Comparing components of eqn.(4.3.27) with eqns.(4.3.24) to (4.3.26), yields the kernels of L^*, as

$$L_1(m,z) = \frac{J_1(m,z)}{\{1+ K_1(z)J_1(z)\}} \tag{4.3.28}$$

$$L_2(m,z_1,z_2) = \left[\frac{J_2(m,z_1,z_2) - J_1(m,z_1z_2)K_2(z_1,z_2)J_1(z_1)J_1(z_2)}{\{1+ K_1(z_1z_2)J_1(z_1z_2)\}}\right] \times \prod_{p=1}^{2} \frac{1}{\{1+ K_1(z_p)J_1(z_p)\}} \tag{4.3.29}$$

and

$$L_3(m,z_1,z_2,z_3) = \left[\frac{J_3(m,z_1,z_2,z_3)}{\prod_{p=1}^{3}\{1+K_1(z_p)J_1(z_p)\}} - \frac{2J_2(m,z_1,z_2z_3)}{\{1+K_1(z_1)J_1(z_1)\}}\{K_1(z_2z_3)L_2(z_2,z_3) + K_2(z_2,z_3)L_1(z_2)L_1(z_3)\}\right.$$
$$\frac{\left. -J_1(m,z_1z_2z_3)\{K_3(z_1,z_2,z_3)\prod_{p=1}^{3} L_1(z_p) + 2K_2(z_1,z_2z_3)L_1(z_1)L_2(z_2,z_3)\}\right]}{\{1 + K_1(z_1z_2z_3)\ J_1(z_1z_2z_3)\}}$$

$$0 < m < 1 \tag{4.3.30}$$

where $L_1(m,z)$ and $L_2(m,z_1,z_2)$ are given by eqns.(4.3.28) and (4.3.29), respectively. The higher order kernels may be similarly obtained. For the synchronously sampled case, the kernels $L_1(z)$, $L_2(z_1,z_2)$ and $L_3(z_1,z_2,z_3)$ may be obtained by letting m=0 in eqns.(4.3.28) to (4.3.30).

However, if the subsystems J and K are not separated by a sampler, then the feedback signal r is given by the operation

$$r = K \otimes y = (K \otimes J) \circledast e^* = M \circledast e^* \tag{4.3.33}$$

where the components $M_1(s)$, $M_2(s_1,s_2)$ and $M_3(s_1,s_2,s_3)$ of M may be obtained from the cascade operation $M = K \otimes J$. The signal r^* is then given by the nonlinear system operation

$$r^* = M^* \circledast e^* \tag{4.3.34}$$

which gives the components of R, in z transforms, as

$$R_1(z) = M_1(z)E_1(z)$$

$$R_2(z_1,z_2) = M_1(z_1z_2)E_2(z_1,z_2) + M_2(z_1,z_2)E_1(z_1)E_1(z_2) \tag{4.3.35}$$

$$R_3(z_1,z_2,z_3) = M_1(z_1z_2z_3)E_3(z_1,z_2,z_3) + 2M_2(z_1,z_2z_3)E_1(z_1)E_2(z_2,z_3) + M_3(z_1,z_2,z_3)\prod_{p=1}^{3} E_1(z_p)$$

where $M_1(z)$, $M_2(z_1,z_2)$ and $M_3(z_1,z_2,z_3)$ may be obtained from $M_1(s)$, $M_2(s_1,s_2)$ and $M_3(s_1,s_2,s_3)$, respectively, using the sequential process of Chapter 2. Substituting eqns.(4.3.35) into eqns.(4.3.22) for components of R and solving gives the components of E in terms of the components of M and the input $U_1(z)$. Then, substituting the resulting equations into the components of eqn.(4.3.23) for components of E and then comparing the resulting equations with the components of eqn. (4.3.27), yields the kernels of L , the equivalent open-loop system representing the given feedback system. The results of this system are given in Appendix A.4. The analysis of nonlinear feedback systems with various locations of samplers is also included in Appendix A.4. The results of all these basic operations may be used now to obtain the Volterra functional expansions of various system connections.

4.4 Example - Feedback Nonlinear System

To illustrate the application of the operations, consider the system shown in Fig.4.7(a), for which the higher order z transform kernels are derived using the operations described in section 4.3. The feedback system is represented in a more convenient form in Fig.4.7(b), in which

$$J_1(z) = \frac{(1-e^{-aT})}{a(z-e^{-aT})} \quad \text{and} \quad K_1(z) = \frac{(1-e^{-bT})}{b(z-e^{-bT})} \tag{4.4.1}$$

α and β are the coefficients of the second and third-order nonlinearities, respectively. The zero-order component in all signals is assumed to be zero, for convenience. All signals are assumed to be characterised by

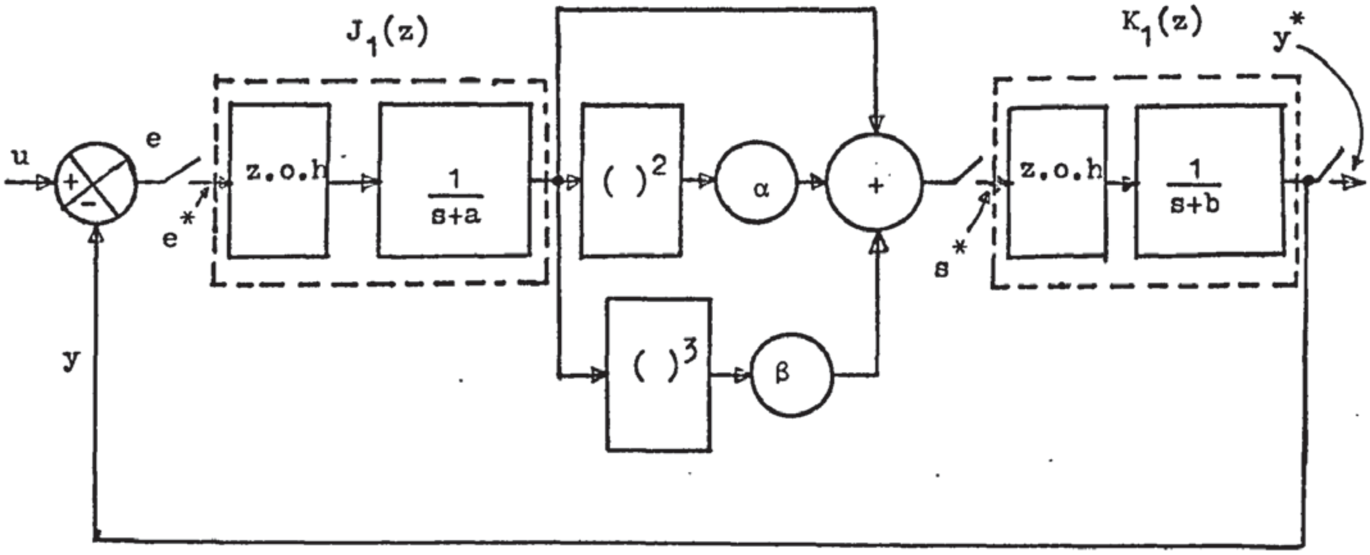


Fig.4.7(a) Discrete nonlinear feedback system.

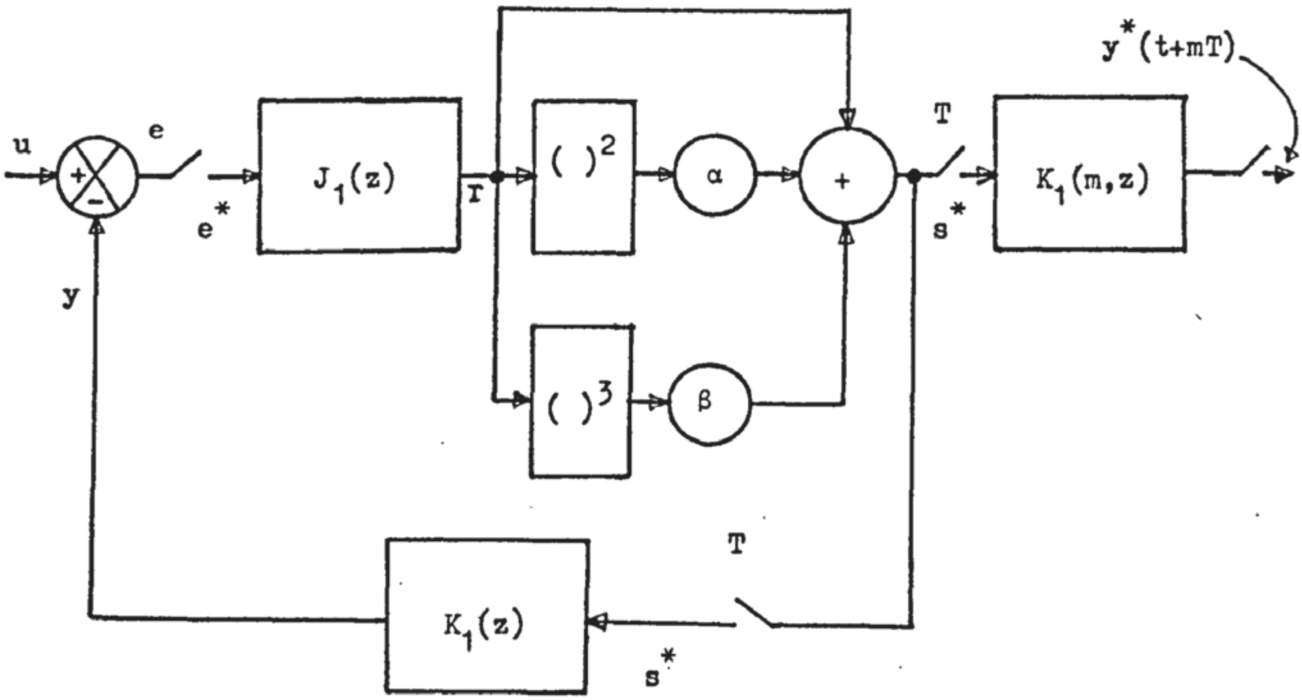


Fig.4.7(b) A convenient form of representing system shown in Fig.4.7(a).

the first three terms of their Volterra series expansions. The components of error signal e^* may be obtained from the summing operation

$$e^* = u^* - y^* \tag{4.4.2}$$

Since the nonlinearities are memoryless, the sampler may well be taken before the nonlinearities. Then, the components of R may be obtained from the linear system operation

$$r^* = J_1^* \otimes e^* \tag{4.4.3}$$

The components of signal s^* may then be obtained from r^* by the multiplication operation,

$$s^* = r^* + \alpha r^* . r^* + \beta r^* .(r^* . r^*) \tag{4.4.4}$$

The output signal y^* may then be obtained from s^* using the linear system operation,

$$y^* = K_1^* \otimes s^* \tag{4.4.5}$$

Using eqns.(4.4.2) and (4.4.3) in eqn.(4.4.4), gives the components of S in terms of components of Y. Then, substituting the resulting equations into components of eqn.(4.4.5) for components of S, and solving for components of Y and then comparing these components with those of eqn. (4.3.27), yields the M.D.M.Z.T kernels of the equivalent system L as

$$L_1(m,z) = \frac{K_1(m,z)J_1(z)}{\{1+ K_1(z)J_1(z)\}}$$

$$L_2(m,z_1,z_2) = \frac{\alpha K_1(m,z_1z_2)}{\{1+ K_1(z_1z_2)J_1(z_1z_2)\}} \prod_{p=1}^{2} \frac{J_1(z_p)}{\{1+ K_1(z_p)J_1(z_p)\}} \tag{4.4.6}$$

$$L_3(m,z_1,z_2,z_3) = \frac{K_1(m,z_1z_2z_3)}{\{1+ K_1(z_1z_2z_3)J_1(z_1z_2z_3)\}}\left[\beta \prod_{p=1}^{3} \frac{J_1(z_p)}{\{1+ K_1(z_p)J_1(z_p)\}} - \frac{2\alpha J_1(z_1)J_1(z_2z_3)L_2(z_2,z_3)}{\{1+ K_1(z_1)J_1(z_1)\}}\right]$$

where $K_1(m,z)$ is given by

$$K_1(m,z) = \frac{z(1-e^{-bmT}) - (e^{-bT}-e^{-bmT})}{b(z-e^{-bT})} \qquad (4.4.7)$$

and $J_1(z)$ and $K_1(z)$ are given by eqns.(4.4.1). The higher degree kernels may be similarly obtained. It should be noted that the above kernels may also be obtained directly from the eqns.(4.3.28) to (4.3.30) of the feedback system operation. It may be noted that if the coefficients α and β of the second and third-order nonlinearities, respectively, are equal to zero, then eqn.(4.4.9) represents the equivalent open-loop system of a linear feedback system.

4.5 Conclusions

Explicit input-output relationships have been given for six operations described here, by which the generalised Volterra functional expansions of various combinations of nonlinear sampled-data systems may be obtained in terms of the generalised Volterra functional expansions of their component subsystems. Block diagram manipulations, as in the linear case, may also be carried out using these operations. In particular the feedback system operation described here can be used for the analysis of a feedback system in which the subsystems J and K may or may not be separated by a sampler. In the former case, the nonlinear system operation must be used and in the latter case, the cascade operation must be used. If these operations were not used, the combination of systems becomes highly involved.

CHAPTER 5

DIGITAL SIMULATION OF NONLINEAR SYSTEMS

5.1 Introduction

Although the analysis of nonlinear systems by Volterra functional series has been the subject of extensive investigation, the application of Volterra series in the field of synthesis has been restricted to mainly continuous systems. Wiener[5] suggested a method to expand a non-separable Volterra kernel, characterising a given nonlinear system, in terms of a set of orthogonal Laguerre functions that can be realised as the impulse responses of linear systems. But, this method of synthesis requires, in general, an infinite number of multipliers and hence fails to be a practical method. This difficulty was overcome by Schetzen[90], who proposed a method for the synthesis of a given Volterra kernel using a finite number of multipliers and first-order linear systems. This method was extended by Bansal[91] for the synthesis of oscillating type kernels using second order-linear systems and finite number of multipliers. The canonic forms for the synthesis of Volterra kernels were suggested by Bush[88].

In this chapter, a method for simulating a continuous nonlinear system by means of a discrete system is described. The canonic forms for realisation of second, third and fourth-order discrete Volterra kernels are introduced. A systematic procedure is developed for the synthesis of second and third-order discrete kernels, using first and second-order linear discrete systems and a finite number of multipliers, and is illustrated by synthesising the first three discrete Volterra kernels of a nonlinear sampled-data feedback system.

A method for the digital simulation of a continuous nonlinear system is developed by which a discrete simulator, for a continuous system preceded by a data-hold device, may be obtained and it is indicated how the number of multipliers in the simulator may be reduced. The practical significance of this method is that it enables the continuous nonlinear system to be simulated on a digital computer, which is often required in dynamic system investigation. The method of obtaining a discrete simulator for a continuous nonlinear system cascaded with a data-hold device is then illustrated by means of three examples.

5.2 Realisation of Nonlinear Discrete System Kernels with a Finite Number of Discrete Linear Systems and Multipliers

A practical problem in the application of discrete Volterra series to nonlinear discrete systems is the realisation of a set of discrete Volterra kernels characterising the given system. When the nonlinearity is small, the given system may be characterised by a finite number of Volterra kernels. For most of the practical systems, a set of first three or four Volterra kernels is adequate, depending on whether the nonlinearities are of odd-order or even-order, respectively. It should be noted that the first-order kernel represents the conventional pulse transfer function of the linearised system and hence may be synthesised by the techniques well known in linear system theory[83]. In this section, canonic or basic forms of second, third and fourth-order kernels are considered. The structures are canonic, not in minimal sense, but in the sense that any realisation by finite number of multipliers and linear discrete systems may be placed in these forms.

5.2.1 Canonic Form of a Second-Order Kernel

A canonic form of a second-order discrete Volterra kernel consisting of three linear discrete systems ${}_1J_1$, ${}_2J_1$ and ${}_3J_1$ and one multiplier is shown in Fig.5.1, in which ${}_iJ_1(z)$, for i=1 and 2, may be first or second-order linear pulse transfer functions given by

$$ {}_iJ_1(z) = \frac{(d_{i1}\ z\ e^{-d_{i2}T} - d_{i3}\ e^{-d_{i4}T})}{(z - e^{-a_iT})} \text{ or } \frac{z\{z - e^{-a_iT}(\cos b_iT + c_i \sin b_iT)\}}{\{z^2 - 2z\ e^{-a_iT}\cos b_iT + e^{-2a_iT}\}}, \tag{5.2.1}$$

respectively and ${}_iJ_1(m,z)$, for i=3, may be a first or second- order linear system with modified z transfer function of the form

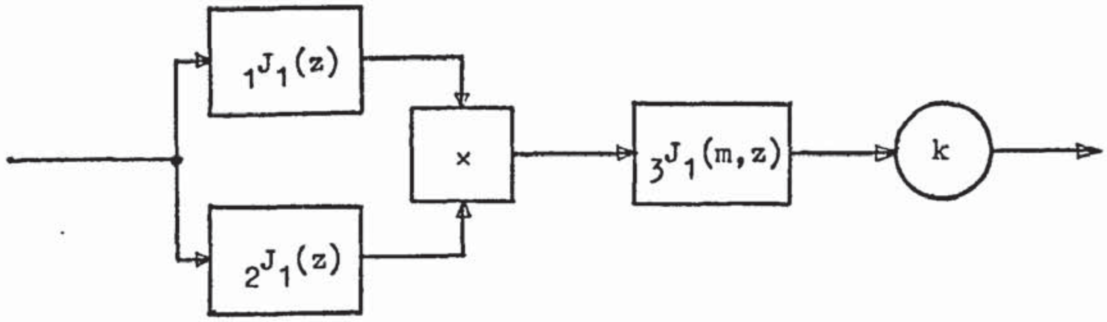


Fig.5.1 Canonic form of a second-order discrete Volterra kernel.

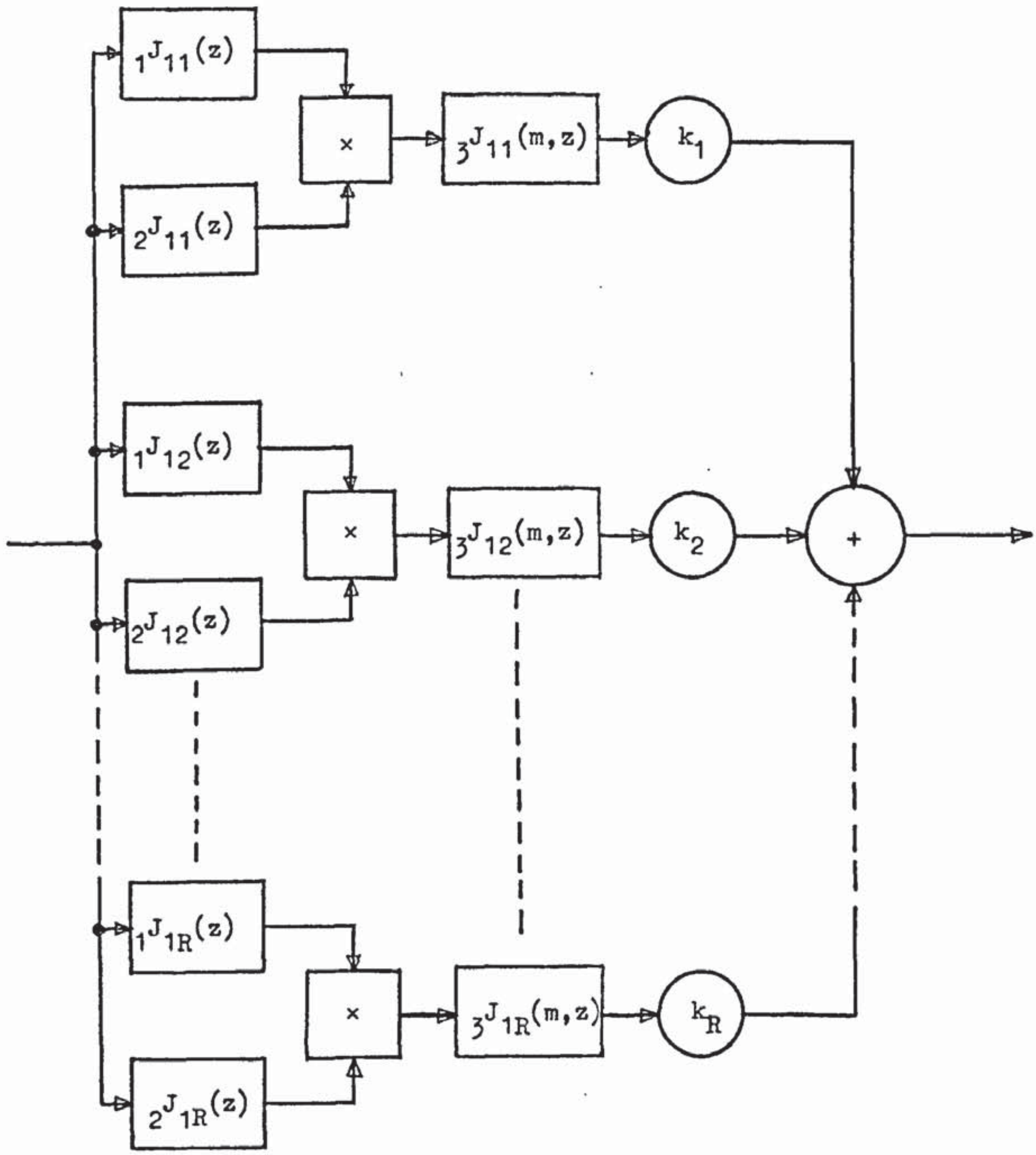


Fig.5.2 A most general second-order kernel.

$$ {}_iJ_1(m,z) = \frac{e^{-d_{i1}mT}(d_{i2}\, z\, e^{-d_{i3}T} - d_{i4}\, e^{-d_{i5}T})}{(z - e^{-a_iT})} \quad \text{or} $$

$$ \frac{ze^{-a_imT}\left[z(\cos mb_iT + c_i\sin mb_iT) - e^{-a_iT}\{\cos(1-m)b_iT - c_i\sin(1-m)b_iT\}\right]}{(z^2 - 2z\, e^{-a_iT}\cos b_iT + e^{-2a_iT})} \tag{5.2.2} $$

respectively, where d_{i1} to d_{i5} are constants(system parameters). Then, $P_2(m,z_1,z_2)$ of the canonic form of the second-order kernel is given by

$$ P_2(m,z_1,z_2) = k\; {}_3J_1(m,z_1z_2)\,{}_2J_1(z_2)\,{}_1J_1(z_1) \tag{5.2.3} $$

where ${}_1J_1(z)$, ${}_2J_1(z)$ and ${}_3J_1(m,z)$ are given by eqns.(5.2.1) and (5.2.2), respectively, and k is a constant. $P_2(m,z_1,z_2)$ may be expressed in the form

$$ P_2(m,z_1,z_2) = \frac{M_2(m,z_1,z_2)}{N_2(z_1,z_2)} = \frac{k\; {}_3M_1(m,z_1z_2)\,{}_2M_1(z_2)\,{}_1M_1(z_1)}{{}_3N_1(z_1z_2)\,{}_2N_1(z_2)\,{}_1N_1(z_1)} \tag{5.2.4} $$

A most general second-order discrete Volterra kernel may then be represented by the sum of the canonic forms of Fig.5.1 and is shown in Fig.5.2, which may be formed by using 3R linear discrete systems and R multipliers. The M.D.M.Z.T of the most general second-order kernel is then given by

$$ L_2(m,z_1,z_2) = \sum_{i=1}^{R} {}_iP_2(m,z_1,z_2) = \sum_{i=1}^{R} \frac{{}_iM_2(m,z_1,z_2)}{{}_iN_2(z_1,z_2)} $$

$$ = \sum_{i=1}^{R} k_i\; {}_3J_{1i}(m,z_1z_2)\,{}_2J_{1i}(z_2)\,{}_1J_{1i}(z_1) \tag{5.2.5} $$

The above equation may be written in the form of eqn.(5.2.4), as

$$ L_2(m,z_1,z_2) = \sum_{i=1}^{R} \frac{k_i\; {}_3M_{1i}(m,z_1z_2)\,{}_2M_{1i}(z_2)\,{}_1M_{1i}(z_1)}{{}_3N_{1i}(z_1z_2)\,{}_2N_{1i}(z_2)\,{}_1N_{1i}(z_1)} $$

$$ = \frac{\left[\sum_{i=1}^{R} k_i\; {}_3M_{1i}(m,z_1z_2)\,{}_2M_{1i}(z_2)\,{}_1M_{1i}(z_1) \prod_{\substack{j=1\\ \neq i}}^{R} {}_3N_{1j}(z_1z_2)\,{}_2N_{1j}(z_2)\,{}_1N_{1j}(z_1)\right]}{\prod_{i=1}^{R} {}_3N_{1i}(z_1z_2)\,{}_2N_{1i}(z_2)\,{}_1N_{1i}(z_1)} \tag{5.2.6} $$

Eqn.(5.2.6) shows that the denominator polynomial of the most general second-order kernel $L_2(m,z_1,z_2)$ is expressible as the product of three functions: a function of z_1, a function of z_2 and a function of (z_1z_2), and the numerator polynomial of $L_2(m,z_1,z_2)$ is also expressible as the sum of R such products. Thus, if R multipliers are used to synthesise a second-order kernel, then the transform of the given second-order kernel must be of the form of eqn.(5.2.6) and if the given second-order Volterra kernel of the nonlinear discrete system is expressible in the form given by eqn.(5.2.6), for some value of R, then it can be realised with at most 3R linear discrete systems and R multipliers. If the transform of the given second-order kernel cannot be expressed in the form of eqn.(5.2.6), then it cannot be realised exactly with a finite number of multipliers and linear discrete systems.

5.2.2 Canonic Form of a Third-Order Kernel

The structure shown in Fig.5.3(a) represents the canonic form of a third-order Volterra kernel, since all third-order kernels that can be synthesised exactly by means of a finite number of linear discrete systems and multipliers can be represented as a sum of all these canonic forms, as in Fig.5.3(b). The canonic form shown in Fig.5.3(a) consists of 2 multipliers and 5 linear discrete systems, and may be regarded as product of a linear system and a second-order canonic form, followed by a linear system. The linear systems ${}_1J_1(z)$ to ${}_4J_1(z)$ may be of first or second-order, whose transfer functions are given by eqn.(5.2.1), for i=1,2,3 and 4, respectively and the system ${}_5J_1(m,z)$ may also be of first or second order linear transfer function given by eqn.(5.2.2) for i=5. The M.D.M.Z.T of the third-order canonic form is then given by

$$P_3(m,z_1,z_2,z_3) = k\ {}_5J_1(m,z_1z_2z_3)\,{}_4J_1(z_3)\,{}_3J_1(z_1z_2)\,{}_2J_1(z_2)\,{}_1J_1(z_1), \quad (5.2.7)$$

which may be expressible in the form

$$P_3(m,z_1,z_2,z_3) = \frac{M_3(m,z_1,z_2,z_3)}{N_3(z_1,z_2,z_3)}$$

$$= \frac{k\ {}_5M_1(m,z_1z_2z_3)\,{}_4M_1(z_3)\,{}_3M_1(z_1z_2)\,{}_2M_1(z_2)\,{}_1M_1(z_1)}{{}_5N_1(z_1z_2z_3)\,{}_4N_1(z_3)\,{}_3N_1(z_1z_2)\,{}_2N_1(z_2)\,{}_1N_1(z_1)} \quad (5.2.8)$$

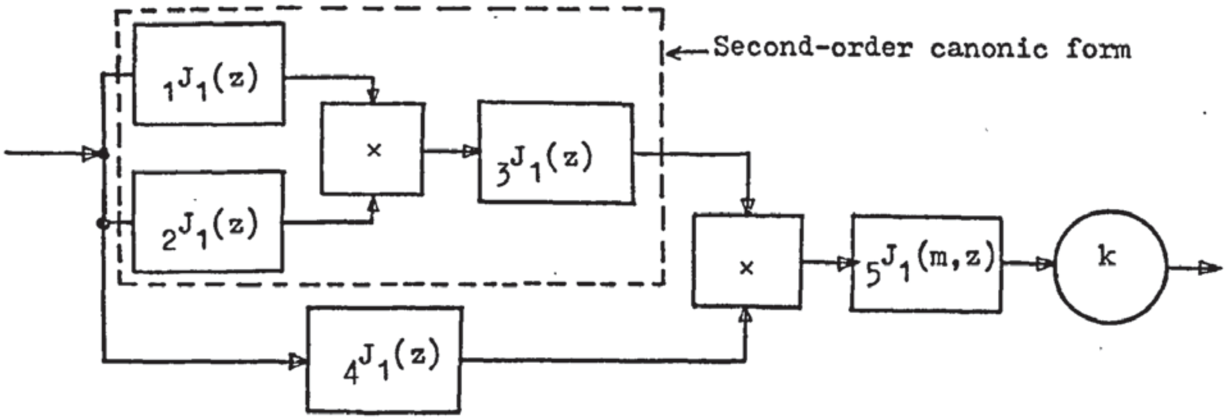


Fig.5.3(a) Canonic form of a third-order kernel.

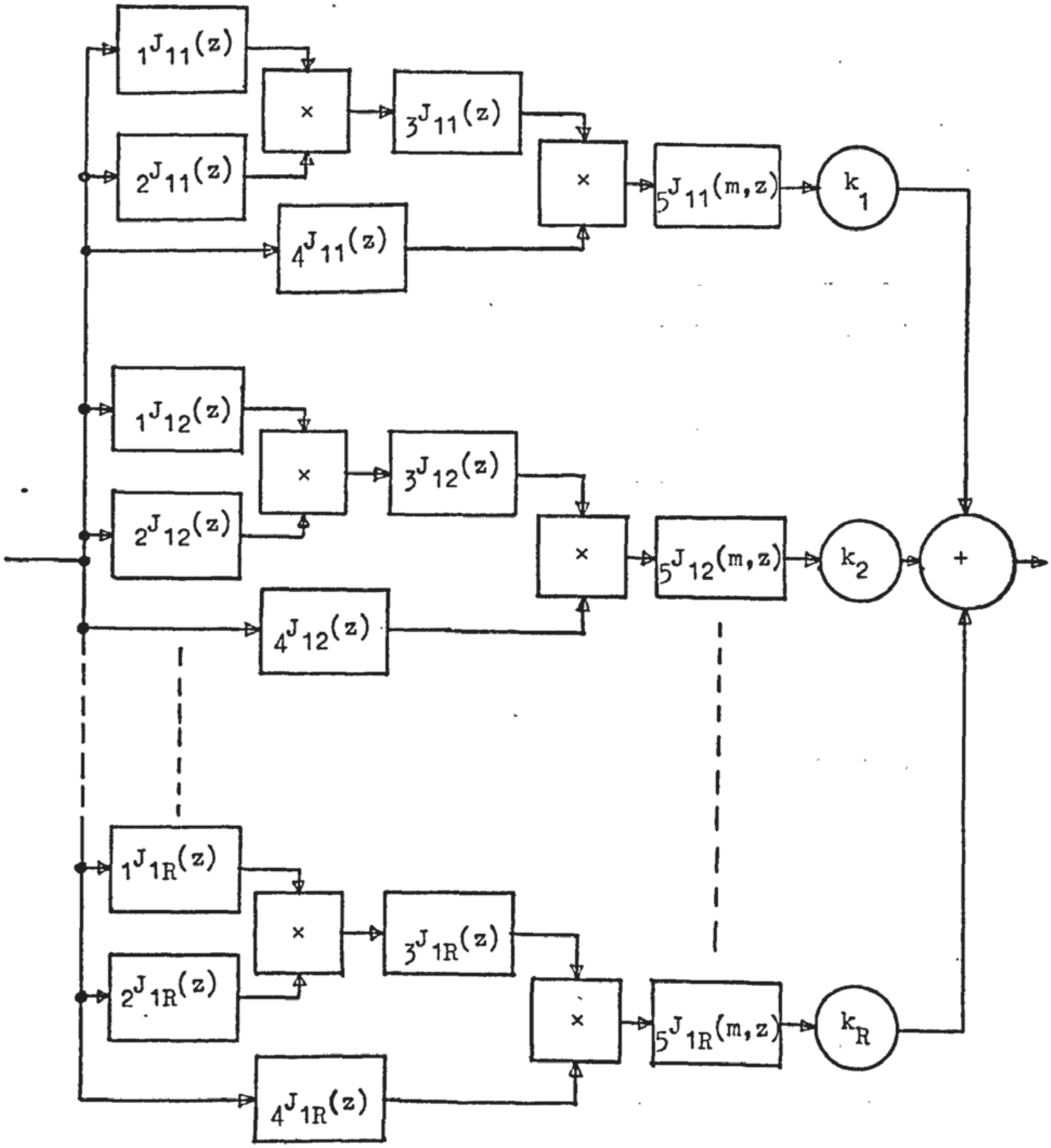


Fig.5.3(b) A most general third-order discrete Volterra kernel.

Thus, the most general third-order kernel that can be synthesised with 5R linear discrete systems and 2R multipliers is given by

$$L_3(m,z_1,z_2,z_3) = \sum_{i=1}^{R} {}_iP_3(m,z_1,z_2,z_3) = \sum_{i=1}^{R} \frac{{}_iM_3(m,z_1,z_2,z_3)}{{}_iN_3(z_1,z_2,z_3)}$$

$$= \sum_{i=1}^{R} k_i\ {}_5J_{1i}(m,z_1z_2z_3)\,{}_4J_{1i}(z_3)\,{}_3J_{1i}(z_1z_2)\,{}_2J_{1i}(z_2)\,{}_1J_{1i}(z_1) \tag{5.2.9}$$

Eqn.(5.2.9) may be expressible as a ratio of polynomials in z variables

$$L_3(m,z_1,z_2,z_3) = \sum_{i=1}^{R} \frac{k_i\ {}_5M_{1i}(m,z_1z_2z_3)\,{}_4M_{1i}(z_3)\,{}_3M_{1i}(z_1z_2)\,{}_2M_{1i}(z_2)\,{}_1M_{1i}(z_1)}{{}_5N_{1i}(z_1z_2z_3)\,{}_4N_{1i}(z_3)\,{}_3N_{1i}(z_1z_2)\,{}_2N_{1i}(z_2)\,{}_1N_{1i}(z_1)}$$

$$= \frac{\left[\sum_{i=1}^{R} k_i\ {}_5M_{1i}(m,z_1z_2z_3)\,{}_4M_{1i}(z_3)\,{}_3M_{1i}(z_1z_2)\,{}_2M_{1i}(z_2)\,{}_1M_{1i}(z_1) \times \prod_{\substack{j=1\\ \neq i}}^{R} {}_5N_{1j}(z_1z_2z_3)\,{}_4N_{1j}(z_3)\,{}_3N_{1j}(z_1z_2)\,{}_2N_{1j}(z_2)\,{}_1N_{1j}(z_1)\right]}{\prod_{i=1}^{R} {}_5N_{1i}(z_1z_2z_3)\,{}_4N_{1i}(z_3)\,{}_3N_{1i}(z_1z_2)\,{}_2N_{1i}(z_2)\,{}_1N_{1i}(z_1)} \tag{5.2.10}$$

Here, the transfer function of each linear system is expressed as a ratio of polynomials. Thus, if and only if, the transform of a given third-order kernel is expressible in the form given by eqn.(5.2.10), for some value of R, then it can be realised with at most 5R linear systems and 2R multipliers, and if the kernel transform cannot be expressed in the form of eqn.(5.2.10), then it is not possible to realise the kernel exactly with a finite number of multipliers and linear discrete systems.

The above results may be easily extended to higher-order kernels, but then the complexity of the procedure increases very rapidly with the order of the kernel. This is because, for an increasing n, there are increasingly more basic forms of structures upto $(n-1)^{th}$-order, to be considered for an n^{th}-order kernel. For example, there are two canonic forms for the fourth-order kernel and are treated in next section.

5.2.3 Canonic Forms of Fourth-Order Kernel

For the fourth-order kernel, there are two canonic forms as shown in Figs.5.4(a) and (b), respectively. Each of them is composed of seven

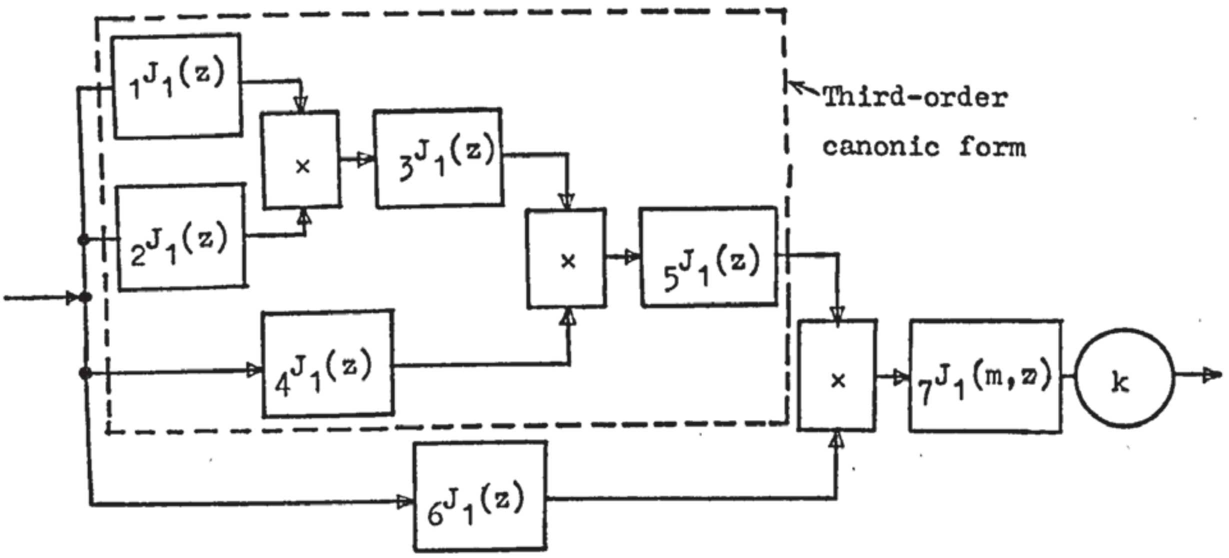


Fig.5.4(a) First canonic form of fourth-order discrete Volterra kernel.

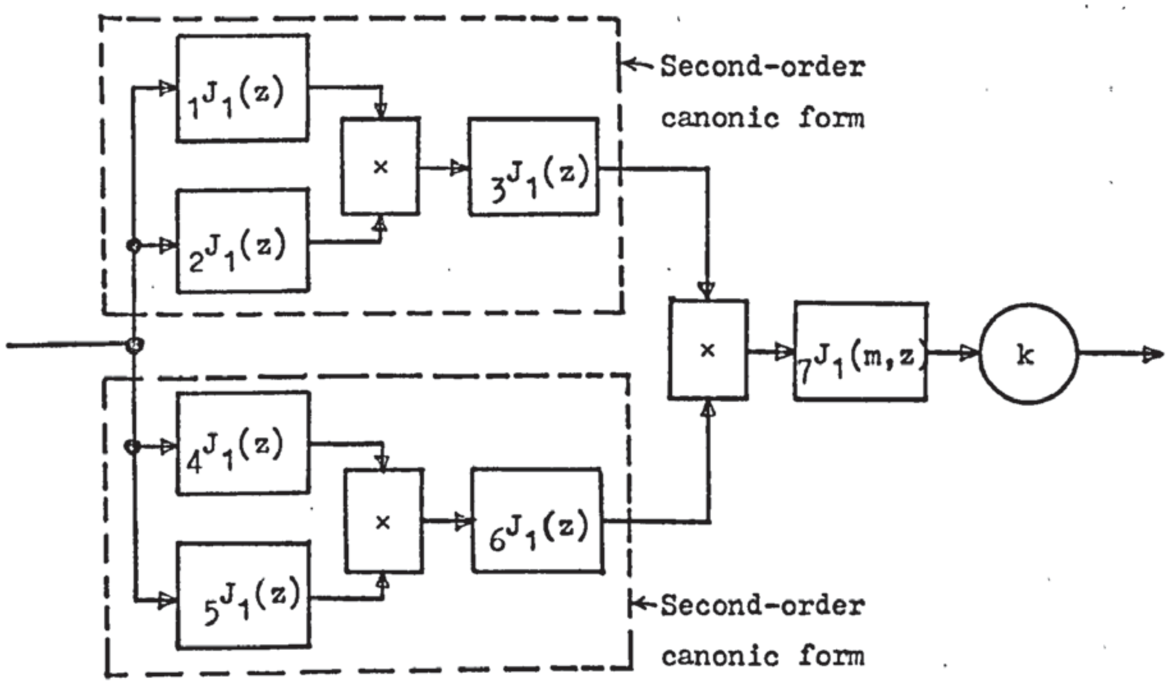


Fig.5.4(b) Second canonic form of fourth-order kernel.

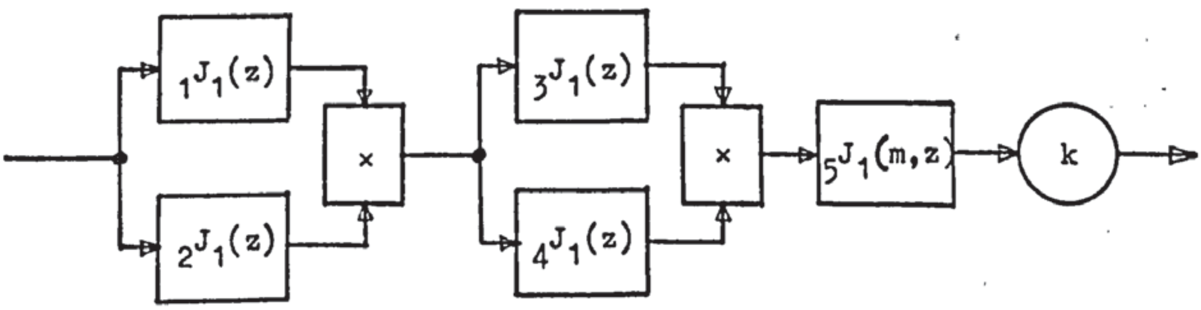


Fig. 5.4(c) Fourth-order kernel with only two multipliers.

linear systems and three multipliers. These two basic forms are essentially different because no block-diagram manipulation can reduce one of these forms to the other. It is clear that the first canonic form may be regarded as a product of a first-order kernel and a third-order canonic form, followed by a linear system, while the second canonic form of the fourth-order kernel is the product of two second-order canonic forms, followed by a linear system. In Figs.5.4(a) and (b), ${}_1J_1(z)$ to ${}_6J_1(z)$ may be first or second order linear transfer functions given by eqn.(5.2.1), for i=1 to 6, respectively, and ${}_7J_1(m,z)$ is a first or second order linear system whose pulse transfer function is given by eqn.(5.2.2), for i=7. The M.D.M.Z.T of the first fourth-order canonic form is given by

$$P_{41}(m,z_1,z_2,z_3,z_4) = {}_7J_1(m,z_1z_2z_3z_4)\,{}_6J_1(z_4)\,{}_5J_1(z_1z_2z_3)\,{}_4J_1(z_3)\,{}_3J_1(z_1z_2) \times {}_2J_1(z_2)\,{}_1J_1(z_1) \qquad (5.2.11)$$

and the second fourth-order canonic form has a transform

$$P_{42}(m,z_1,z_2,z_3,z_4) = {}_7J_1(m,z_1z_2z_3z_4)\,{}_6J_1(z_3z_4)\,{}_5J_1(z_4)\,{}_4J_1(z_3)\,{}_3J_1(z_1z_2) \times {}_2J_1(z_2)\,{}_1J_1(z_1) \qquad (5.2.12)$$

The most general fourth-order kernel with 7R linear systems and 3R multipliers may therefore be composed of R_1 first canonic forms and R_2 second canonic forms, where $R=(R_1+R_2)$. Thus, if 3R multipliers are used to synthesise a given fourth-order kernel, then its transform must be of the form

$$L_4(m,z_1,z_2,z_3,z_4) = \sum_{i=1}^{R_1} {}_iP_{41}(m,z_1,z_2,z_3,z_4) + \sum_{i=1}^{R_2} {}_iP_{42}(m,z_1,z_2,z_3,z_4) \qquad (5.2.13)$$

It may be noted that any fourth-order kernel that may be synthesised with three or less number of multipliers, may be arranged into one of these canonic forms. For example, the fourth-order kernel shown in Fig.5.4(c) can be arranged in the canonic form of Fig.5.4(b). If the transform of a given fourth-order kernel cannot be expressed in the form of (5.2.13), then it cannot be realised exactly with a finite number of multipliers and linear discrete systems.

5.3 A Procedure for Synthesis of a Discrete Volterra Kernel

A Volterra kernel of a given nonlinear discrete system may be synthesised by a finite number of linear discrete systems and multipliers, if and only if, its transform is expressible in the form given by eqn. (5.2.6) for second-order kernels, eqn.(5.2.10) for third-order kernels and eqn.(5.2.13) for fourth-order kernels. A systematic procedure for the synthesis of second and third-order kernels is given here and may be easily extended to fourth and higher-order kernels, if required.

5.3.1 Synthesis of a Second-Order Kernel

The systematic procedure for the synthesis of a given second-order kernel is outlined in Table 5.1. Let the transform of a given kernel be expressible in the form

$$P_2(m,z_1,z_2) = \frac{M_2(m,z_1,z_2)}{N_2(z_1,z_2)} \tag{5.3.1}$$

The tests carried out, in Table 5.1, on the numerator provides an information regarding the order of the linear discrete systems ${}_1J_1(z)$, ${}_2J_1(z)$ and ${}_3J_1(m,z)$. It should be noted, however, that if the systems ${}_1J_1(z)$ and ${}_2J_1(z)$ have same zeros, as in the case of quadratic nonlinear systems, then the test 1 fails. This difficulty can be overcome by computing $M_2(m,z,z^{-1})$, which gives ${}_3M_1(m,z)$, if it exists, as a constant. This is demonstrated in the following example. The tests carried out on the denominator will confirm the results of the tests of the numerator, and also help in realising the system parameters. Finally, the value of the constant k can be determined, by inspection.

5.3.2 Synthesis of a Third-Order Kernel

The procedure described in Table 5.1 may be extended to synthesise third and higher-order kernels. Let the given third-order kernel transform be expressible in the form

$$P_3(m,z_1,z_2,z_3) = \frac{M_3(m,z_1,z_2,z_3)}{N_3(z_1,z_2,z_3)} \tag{5.3.2}$$

The procedure for synthesis of the kernel $P_3(m,z_1,z_2,z_3)$ is given in

Table 5.1 Synthesis of a Second-Order Kernel :

No.	Test	Result	Conclusion
1.	Test $M_2(m,1,z)$ and $M_2(m,z,1)$ for a common factor.	A common factor of the form $_3M_1(m,z) = ze^{-a_3mT}\left[zf_1(m,b_3) - e^{-a_3T}f_2(m,b_3,c_3)\right]$, exists.	$_3J_1(m,z)$ is a second-order linear system, and $f_1(m,b_3)=\{\cos mb_3T + c_3\sin mb_3T\}$ $f_2(m,b_3,c_3)=\{\cos(1-m)b_3T - c_3\sin(1-m)b_3T\}$ (T.5.1)
		A common factor of the form $_3M_1(m,z) = e^{-d_{31}mT} x (d_{32}ze^{-d_{33}T} - d_{34}e^{-d_{35}T})$, exists.	$_3J_1(m,z)$ is a first or second-order linear system and d_{31} to d_{35} are constants determined by comparison.
		No such common factor exists.	$_3J_1(m,z)$ is either absent or a first or second-order linear system.
	Compute $M_2(m,z,z^{-1})$	$_3M_1(m,z)$ is a constant.	$_3J_1(m,z)$ is a first or second-order linear system.
2.	Divide $M_2(m,z_1,z_2)$ by $_3M_1(m,z_1z_2)$, if it exists, to give $D_2(z_1,z_2)$. Test $D_2(z,1)$ for a factor.	A factor of the form $_1M_1(z) = z\left[z - e^{-a_1T}f_3(b_1,c_1)\right]$ exists.	$_1J_1(z)$ is a second-order linear system and $f_3(b_1,c_1)=\{\cos b_1T - c_1\sin b_1T\}$ (T.5.2)
		A factor of the form $_1M_1(z)=(d_{11}ze^{-d_{12}T} - d_{13}e^{-d_{14}T})$, exists.	$_1J_1(z)$ is a first or second-order system and d_{11} to d_{14} are determined by comparison.
		No such factor exists.	$_1J_1(z)$ is either absent or a first or second-order system.
3.	Divide $D_2(z_1,z_2)$	$_2M_1(z) = z\{z - e^{-a_2T}f_4(b_2,c_2)\}$, exists.	$_2J_1(z)$ is a second-order linear system and $f_4(b_2,c_2)=(\cos b_2T - c_2\sin b_2T)$ (T.5.3)

Table 5.1 (contd.)

	by $_1M_1(z_1)$, if it exists in step 2, to give $_2M_1(z_2)$.	$_2M_1(z)=(d_{21}ze^{-d_{22}T} - d_{23}e^{-d_{24}T})$, exists.	$_2J_1(z)$ is a first or second-order system and d_{21} to d_{24} are determined by comparison.
		$_2M_1(z)$ is a constant.	$_2J_1(z)$ is either absent or a first or second order system.
4.	Test $N_2(1,z)$ and $N_2(z,1)$ for a common factor.	A common factor of the form $_3N_1(z)= (z^2- 2ze^{-a_3T}\cos b_3T + e^{-2a_3T})$, exists.	$_3J_1(m,z)$ is a second-order linear system. Knowing the values of a_3 and b_3, the constant c_3 can be obtained from eqn.(T.5.1) and $_3J_1(m,z)$ is realised.
		A common factor of the form $_3N_1(z)=(z- e^{-a_3T})$ exists.	$_3J_1(m,z)$ is a first-order linear system and can be realised by knowing a_3.
		No such common factor exists.	$_3J_1(m,z)$ is absent.
	Compute $N_2(z,z^{-1})$, if necessary.	$_3N_1(z)$ is a constant.	
5.	Divide $N_2(z_1,z_2)$ by $_3N_1(z_1z_2)$, if it exists in step 4, to give $F_2(z_1,z_2)$. Test $F_2(1,z)$ for a factor.	A factor of the form $_2N_1(z) =(z^2-2z\,e^{-a_2T}\cos b_2T + e^{-2a_2T})$, exists.	$_2J_1(z)$ is a second-order linear system. Knowing a_2 and b_2, c_2 can be determined from eqn.(T.5.3) and hence $_2J_1(z)$ may be realised completely.
		A factor of the form $_2N_1(z) =(z- e^{-a_2T})$, exists.	$_2J_1(z)$ is a first-order linear system and can be realised by knowing a_2.
		$F_2(1,z)$ is a constant.	$_2J_1(z)$ is absent.
6.	Divide $F_2(z_1,z_2)$ by $_2N_1(z_2)$, if it exi- in step 5,	$_1N_1(z)=(z^2-2ze^{-a_1T}\cos b_1T + e^{-2a_1T})$, exists.	$_1J_1(z)$ is a second-order linear system. Knowing a_1 and b_1, c_1 may be determined from eqn.(T.5.2) and hence $_1J_1(z)$ can be realised.

Table 5.1 (contd.)

	to give ${}_1N_1(z_1)$	${}_1N_1(z)=(z- e^{-a_1T})$, exists. ${}_1N_1(z)$ is a constant.	${}_1J_1(z)$ is a first-order linear system and can be realised by knowing a_1. ${}_1J_1(z)$ is absent.
7.	$\dfrac{M_2(m,z_1,z_2)}{N_2(z_1,z_2)}$	$\dfrac{k\ {}_3M_1(m,z_1z_2)_2M_1(z_2)_1M_1(z_1)}{{}_3N_1(z_1z_2)_2N_1(z_2)_1N_1(z_1)}$	Comparing both, yields the constant k.

Appendix A.5.

For the synthesis of the kernels of a nonlinear feedback discrete system shown in Fig.4.6(a), the following procedure may be used: The explicit input-output kernels $L_1(m,z)$ to $L_3(m,z_1,z_2,z_3)$ characterising a given nonlinear feedback system may be obtained, in terms of the kernels of the subsystems J and K, from eqns.(4.3.28) to (4.3.30) of Chapter 4, respectively. Now, the kernels $L_2(m,z_1,z_2)$ and $L_3(m,z_1,z_2,z_3)$ can be realised using the synthesis procedure described in Table 5.1 and Appendix A.5, respectively. This is illustrated in the following example. Regarding the realisability of the kernels of a feedback system, it may be pointed out that if each of the kernels of the subsystems J and K is realisable exactly by a finite number of multipliers and first-order linear discrete systems, then each of the kernels L_1 to L_3 of the feedback system is also exactly realisable, but with a finite number of second-order linear discrete systems and multipliers.

5.3.3 Example - Feedback Nonlinear System

The method of synthesis developed here is illustrated by synthesising the Volterra kernels of a sampled-data nonlinear feedback system shown in Fig.4.7, of Chapter 4, for which the first three kernels are given by eqns. (4.4.6a) to (4.4.6c), respectively. $L_1(m,z)$ is a linear kernel and is realised as shown in Fig.5.5. The second-order kernel $L_2(m,z_1,z_2)$ given by eqn.(4.4.6b) is in the form

$$L_2(m,z_1,z_2) = \sum_{i=1}^{3} {}_iP_2(m,z_1,z_2) = \sum_{i=1}^{3} \frac{{}_iM_2(m,z_1,z_2)}{{}_iN_2(z_1,z_2)} \tag{5.3.3}$$

where

$$\begin{aligned} {}_1M_2(m,z_1,z_2) = k_1\{&z_1^3z_2^3(1-e^{-bmT}) - z_1^3z_2^2e^{-bT}(1-e^{-bmT}) - z_1^2z_2^3e^{-bT}(1-e^{-bmT}) \\ &+ z_1^2z_2^2e^{-2bT}(1-e^{-bmT})\} \end{aligned}$$

$$\left.\begin{aligned} {}_2M_2(m,z_1,z_2) &= -k_2(z_1^2z_2^2 - z_1^2z_2\, e^{-bT} - z_1z_2^2\, e^{-bT} + z_1z_2e^{-2bT})\alpha_3 \\ {}_3M_2(m,z_1,z_2) &= k_3(z_1z_2\alpha_4 - z_1\alpha_4e^{-bT} - z_2\alpha_4e^{-bT} + \alpha_4e^{-2bT}) \end{aligned}\right\} \tag{5.3.4}$$

$$\begin{aligned} {}_1N_2(z_1,z_2) &= {}_2N_2(z_1,z_2) = {}_3N_2(z_1,z_2) \\ &= \{z_1^4z_2^4 - z_1^4z_2^3\alpha_1 + z_1^4z_2^2\alpha_2 - z_1^3z_2^4\alpha_1 - z_1^3z_2^3\alpha_1(\alpha_1+1) - z_1^3z_2^2\alpha_1(\alpha_2-\alpha_1) + z_1^2z_2^4\alpha_2 \\ &\quad - z_1^2z_2^3\alpha_1(\alpha_2-\alpha_1) + z_1^2z_2^2(\alpha_1^3+\alpha_2^2+\alpha_2) - z_1^3z_2\alpha_1\alpha_2 + z_1^2z_2\alpha_1\alpha_2(\alpha_1-1) - z_1z_2^3\alpha_1\alpha_2 \\ &\quad + z_1z_2^2\alpha_1\alpha_2(\alpha_1-1) - z_1z_2\alpha_1\alpha_2(\alpha_2+\alpha_1) - z_1\alpha_1\alpha_2^2 - z_2\alpha_1\alpha_2^2 + z_1^2\alpha_2^2 + z_2^2\alpha_2^2 + \alpha_2^3\} \end{aligned}$$

where $\alpha_1 = (e^{-aT}+e^{-bT})$, $\alpha_2 = (\lambda + e^{-(a+b)T})$, $\alpha_3 = (e^{-bT}-e^{-aT}) - e^{-bmT}(1-e^{-aT})$,

$$\alpha_4 = e^{-aT}(e^{-bT}-e^{-bmT}),\ \lambda = \frac{(1-e^{-aT})(1-e^{-bT})}{ab} \text{ and } k_1 = k_2 = k_3 = \frac{\alpha(1-e^{-aT})^2}{a^2b} \tag{5.3.5}$$

First, $\dfrac{{}_1M_2(m,z_1,z_2)}{{}_1N_2(z_1,z_2)}$ is synthesised, using the procedure described in Table 5.1.

<u>Step No.1</u>: Testing the numerator ${}_1M_2(m,z_1,z_2)$, for a common factor, gives

$$\left.\begin{aligned} {}_1M_2(m,1,z) &= z^2(1-e^{-bmT})(1-e^{-bT})(z-e^{-bT}) \quad \text{and} \\ {}_1M_2(m,z,1) &= z^2(1-e^{-bmT})(z-e^{-bT})(1-e^{-bT}) \end{aligned}\right\} \tag{5.3.6}$$

The above equations show that both factors $z_1^2z_2^2(1-e^{-bmT})$ and $(z_1z_2-e^{-bT})$ exist as common factors. But, this is not true. To find out the required common factor, compute ${}_1M_2(m,z,z^{-1})$ which gives

$${}_1M_2(m,z,z^{-1}) = (1-e^{-bmT})(z-e^{-bT})(z^{-1}-e^{-bT}) \tag{5.3.7}$$

Now, eqns.(5.3.6) and (5.3.7) show that a common factor of the form ${}_3M_{11}(m,z_1z_2) = z_1^2z_2^2(1-e^{-bmT})$, exists. Hence, ${}_3J_{11}(m,z)$ is a second-order linear system and

$$e^{-a_3mT}f_1(m,b_3) = (1-e^{-bmT}) \text{ and } f_2(m,b_3,c_3) = 0. \tag{5.3.8}$$

Step No.2: Dividing ${}_1M_2(m,z_1,z_2)$ by ${}_3M_{11}(m,z_1z_2)$ gives ${}_1D_2(z_1,z_2) = (z_1z_2 - z_1e^{-bT} - z_2e^{-bT} + e^{-2bT})$. Testing ${}_1D_2(z,1)$ for a factor, gives

$$ {}_1D_2(z,1) = (z - e^{-bT})(1 - e^{-bT}) \tag{5.3.9} $$

Since a factor of the form ${}_1M_{11}(z_1) = (z_1 - e^{-bT})$ exists, ${}_1J_{11}(z)$ may be a first or second-order linear system and the constants are given by

$$ d_{11} = 1,\ d_{12} = 0,\ d_{13} = 1 \text{ and } d_{14} = b. \tag{5.3.10} $$

Step No.3: Dividing ${}_1D_2(z_1,z_2)$ by ${}_1M_{11}(z_1)$ gives ${}_2M_{11}(z_2) = (z_2 - e^{-bT})$. Hence, ${}_2J_{11}(z)$ may be a first or a second-order linear system and the constants d_{21} to d_{24} are given by

$$ d_{21} = 1,\ d_{22} = 0,\ d_{23} = 1 \text{ and } d_{24} = b. \tag{5.3.11} $$

Step No.4: Testing the denominator ${}_1N_2(z_1,z_2)$ gives

$$ \begin{aligned} {}_1N_2(1,z) &= (z^2 - z\alpha_1 + \alpha_2)(1 - \alpha_1 + \alpha_2)(z^2 - z\alpha_1 + \alpha_2) \\ {}_1N_2(z,1) &= (z^2 - z\alpha_1 + \alpha_2)(z^2 - z\alpha_1 + \alpha_2)(1 - \alpha_1 + \alpha_2) \end{aligned} \tag{5.3.12} $$

The above equations show that two common factors of the form $(z_1^2z_2^2 - z_1z_2\alpha_1 + \alpha_2)$ exist and this is not true. To identify the true common factor, compute ${}_1N_2(z,z^{-1})$, which gives

$$ {}_1N_2(z,z^{-1}) = (1 - \alpha_1 + \alpha_2)(z^2 - z\alpha_1 + \alpha_2)(z^{-2} - z^{-1}\alpha_1 + \alpha_2) \tag{5.3.13} $$

Thus, only one common factor of the form ${}_3N_{11}(z_1z_2) = (z_1^2z_2^2 - z_1z_2\alpha_1 + \alpha_2)$, exists. Hence, ${}_3J_{11}(m,z)$ is a second-order linear system and

$$ 2e^{-a_3T}\cos b_3T = \alpha_1 \quad \text{and} \quad e^{-2a_3T} = \alpha_2, \tag{5.3.14} $$

from which a_3 and b_3 can be computed, since α_1 and α_2 are known.

Step No.5: Dividing ${}_1N_2(z_1,z_2)$ by ${}_3N_{11}(z_1z_2)$ gives

$$ {}_1F_2(z_1,z_2) = (z_1^2z_2^2 - z_1^2z_2\alpha_1 + z_1^2\alpha_2 - z_1z_2^2\alpha_1 - z_1z_2\alpha_1^2 - z_1\alpha_1\alpha_2 + z_2^2\alpha_2 - z_2\alpha_1\alpha_2 + \alpha_2^2) \tag{5.3.15} $$

Testing ${}_1F_2(1,z)$ for a factor gives

$$ {}_1F_2(1,z) = (1 - \alpha_1 + \alpha_2)(z^2 - z\alpha_1 + \alpha_2) \tag{5.3.16} $$

A factor of the form ${}_2N_{11}(z_2) = (z_2^2 - z_2\alpha_1 + \alpha_2)$ exists. Hence, ${}_2J_{11}(z)$ is a second-order linear system and the constants a_2 and b_2 can be determined from

$$2e^{-a_2T}\cos b_2T = \alpha_1 \quad \text{and} \quad e^{-2a_2T} = \alpha_2. \tag{5.3.17}$$

Step No.6: Dividing ${}_1F_2(z_1,z_2)$ by ${}_2N_{11}(z_2)$ gives

$${}_1N_{11}(z_1) = (z_1^2 - z_1\alpha_1 + \alpha_2) \tag{5.3.18}$$

By comparison, it is clear that ${}_1J_{11}(z_1)$ is a second-order linear system and the constants a_1 and b_1 can be determined from

$$2e^{-a_1T}\cos b_1T = \alpha_1 \quad \text{and} \quad e^{-2a_1T} = \alpha_2 . \tag{5.3.19}$$

Step No.7: By comparing $\dfrac{{}_1M_2(m,z_1,z_2)}{{}_1N_2(z_1,z_2)}$ and $\dfrac{k_1\,{}_3M_{11}(m,z_1z_2)\,{}_2M_{11}(z_2)\,{}_1M_{11}(z_1)}{{}_3N_{11}(z_1z_2)\,{}_2N_{11}(z_2)\,{}_1N_{11}(z_1)}$, the value of the constant k_1 may be obtained as

$$k_1 = \frac{\alpha(1-e^{-aT})^2}{a^2b}$$

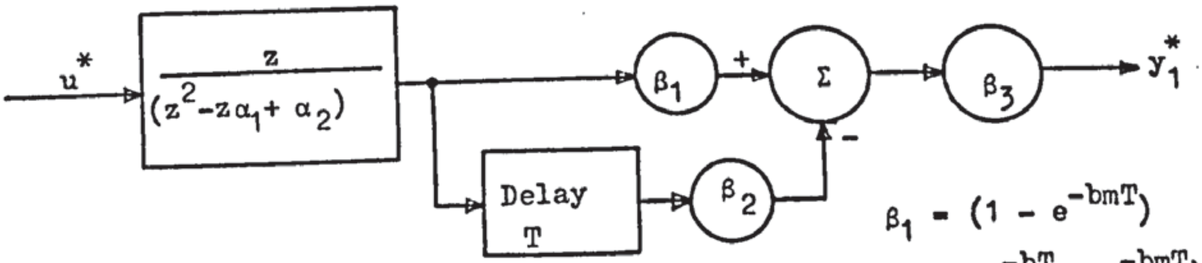
The complete synthesis of the kernel ${}_1P_2(m,z_1,z_2)$ is shown in Fig.5.6(a). Similarly, ${}_2P_2(m,z_1,z_2)$ and ${}_3P_2(m,z_1,z_2)$ are realised as shown in Fig.5.6 (b) and (c), respectively giving the complete second-order simulator shown in Fig.5.6(d), in which $e^{-\lambda_1T}$ and $e^{-\lambda_2T}$ are given by

$$e^{-\lambda_1T},\ e^{-\lambda_2T} = \left[\frac{(e^{-aT}+e^{-bT}) \pm \sqrt{\{(e^{-aT}+e^{-bT})^2 - 4(\lambda + e^{-(a+b)T})\}}}{2}\right]$$

The third-order simulator represented by the kernel $L_3(m,z_1,z_2,z_3)$ can be similarly realised as shown in Fig.5.7, using the procedure described in Appendix A.5. Then, the sampled response of the feedback system, shown in Fig.4.7, may be obtained by adding the outputs of the linear, second-order and third-order simulators, shown in Figs.5.5 to 5.7, respectively.

5.4 Digital Simulation of Continuous Nonlinear Systems

The simulation of a continuous system by means of a discrete system often becomes necessary in dynamic system investigations. For the digital simulation of a continuous system, it is required to apply the sampled-data input u^* to the system through a data-hold device in order



$\beta_1 = (1 - e^{-bmT})$

$\beta_2 = (e^{-bT} - e^{-bmT})$

$\beta_3 = \frac{(1 - e^{-aT})}{ab}$

Fig.5.5 Synthesis of the first-order kernel.

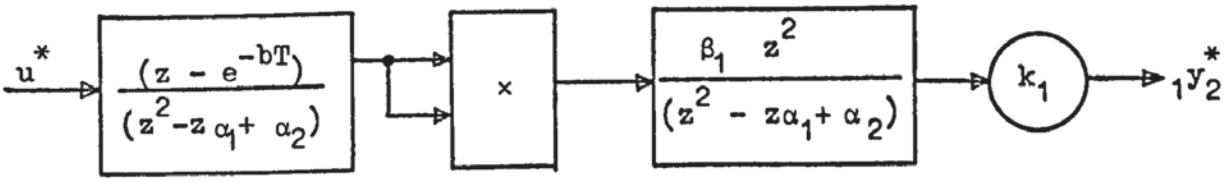


Fig.5.6(a) Synthesis of ${}_1P_2(m,z_1,z_2)$.

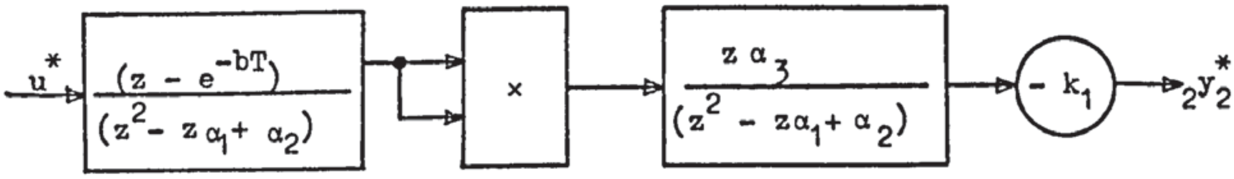


Fig.5.6(b) Synthesis of ${}_2P_2(m,z_1,z_2)$.

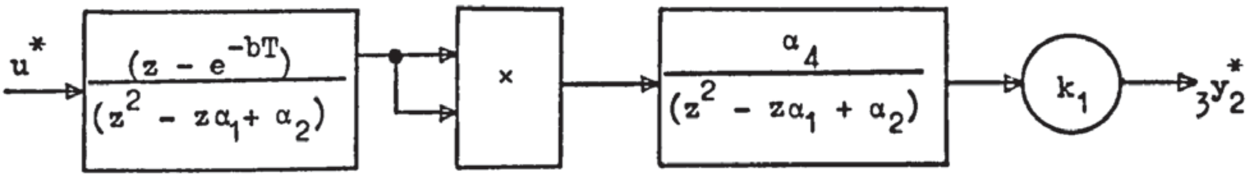


Fig.5.6(c) Synthesis of ${}_3P_2(m,z_1,z_2)$.

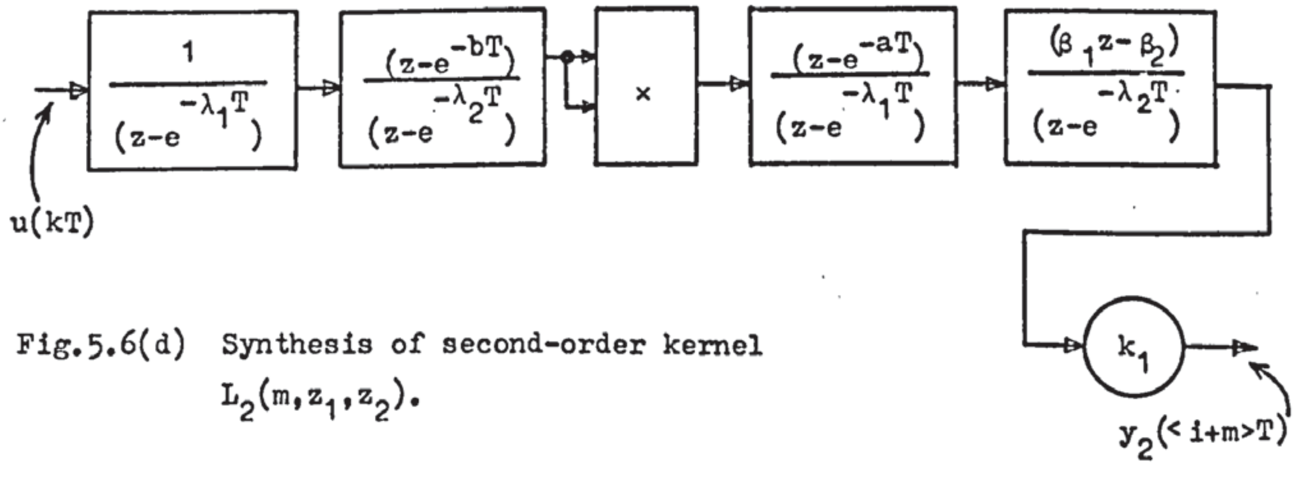


Fig.5.6(d) Synthesis of second-order kernel $L_2(m,z_1,z_2)$.

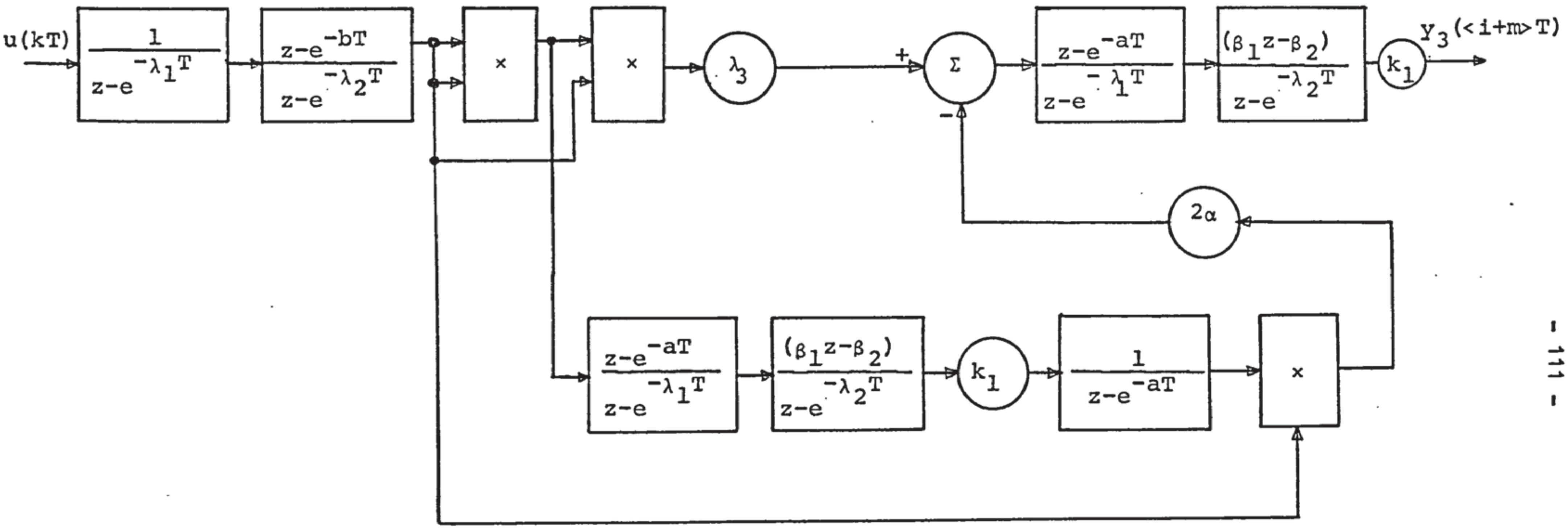


Fig.5.7 Synthesis of the third-order kernel, $L_3(m,z_1,z_2,z_3)$

to reduce the loss of information resulting from the sampling of the input signal. The problem is, therefore, to obtain a discrete simulator for the cascade combination of the data-hold-device and the continuous system. One of the main applications of the synthesis procedures developed for second and third-order kernels, in section 5.3, is in realising the multidimensional z transform kernels characterising a given nonlinear sampled-data system. In this section, the method of synthesising the z transform kernels of a given nonlinear sampled-data system is presented and is demonstrated by means of various examples.

5.4.1 Discrete Simulator for a Continuous System

To obtain a discrete simulator for a continuous nonlinear system, the multidimensional z transform of the cascade combination of a data-hold device and a continuous nonlinear system is first derived using the sequential process of Chapter 2, and then the nonlinear discrete system, which has this multidimensional z transform as its kernel, is synthesised using the procedure developed in section 5.3.

5.4.2 Example 1- Simulation of a Nonlinear System with 2^{nd}-Order Kernel

Consider the continuous nonlinear system with a second-order kernel shown in Fig.5.8(a). An exact solution to the response of the system, for stepwise continuous inputs, may be obtained if the system is simulated as shown in Fig.5.8(b). The multidimensional z transform $L_2(z_1,z_2)$ of the cascade combination of the zero-order-hold and the system, given by eqn.(2.4.4), is expressible in the form, for m=0, as

$$L_2(z_1,z_2) = \sum_{i=1}^{4} {}_iP_2(z_1,z_2) = \sum_{i=1}^{4} \frac{{}_iM_2(z_1,z_2)}{{}_iN_2(z_1,z_2)} \qquad (5.4.1)$$

where $\dfrac{{}_1M_2(z_1,z_2)}{{}_1N_2(z_1,z_2)} = \dfrac{k_1}{(z_1^2z_2 - z_1z_2e^{-aT} - z_1e^{-bT} + e^{-(a+b)T})}$

$$\frac{{}_2M_2(z_1,z_2)}{{}_2N_2(z_1,z_2)} = \frac{k_2}{(z_1z_2^2 - z_1z_2e^{-aT} - z_2e^{-bT} + e^{-(a+b)T})} \qquad (5.4.2)$$

$$\frac{{}_3M_2(z_1,z_2)}{{}_3N_2(z_1,z_2)} = \frac{k_3}{(z_1z_2 - z_1e^{-aT} - z_2e^{-aT} + e^{-2aT})}$$

and $\quad \dfrac{{}_4M_2(z_1,z_2)}{{}_4N_2(z_1,z_2)} = \dfrac{k_4}{(z_1z_2 - e^{-bT})}$

where

$$k_1 = k_2 = \frac{\alpha^2\beta(1-e^{-aT})(e^{-bT}-e^{-aT})}{a(b-a)(b-2a)}\;;\quad k_3 = \frac{\alpha^2\beta(1-e^{-aT})^2}{a^2(b-2a)} \quad \text{and}$$

$$k_4 = \frac{2\,\alpha^2\beta\{a(1-e^{-bT})-b(1-e^{-aT})\}}{ab(b-a)(b-2a)} \tag{5.4.3}$$

Now, each of the kernels ${}_iP_2(z_1,z_2)$, $i=1,2,3,4$, may be synthesised, using the procedure described in Table 5.1, as shown in Figs.5.9(a) to (d), respectively. Then, the discrete simulator for the system shown in Fig.5.8(b) is obtained as shown in Fig.5.10(a). It may be noted that this simulator requires three multipliers and two linear discrete systems. However, by simplifying and rearranging the terms of the multidimensional z transform $L_2(z_1,z_2)$ given by eqn.(2.4.4), an alternative form of the simulator can be realised with only two multipliers and two linear discrete systems, as shown in Fig.5.10(b), for which $L_2(z_1,z_2)$ is given by

$$L_2(z_1,z_2) = \frac{k_5}{(z_1z_2-e^{-bT})} + \frac{k_6}{(z_1-e^{-aT})(z_2-e^{-aT})}$$

$$+\frac{k_7}{(z_1z_2-e^{-bT})(z_1-e^{-aT})(z_2-e^{-aT})} \tag{5.4.5}$$

where $k_5 = \dfrac{\alpha^2\beta\{2a^2(1-e^{-bT})-ab(1-e^{-aT})(1+e^{-(b-a)T})\}}{(b-a)(b-2a)}$,

$$k_6 = \frac{\alpha^2\beta b(1-e^{-aT})}{(b-a)(b-2a)}\{(b-2a)(1-e^{-aT})-a\,e^{-aT}(1-e^{-(b-2a)T})\}, \text{ and}$$

$$k_7 = -\frac{\alpha^2\beta ab(1-e^{-aT})(1-e^{-(b-a)T})(e^{-bT}-e^{-2aT})}{(b-a)(b-2a)} \tag{5.4.6}$$

Thus, the realisation of a discrete simulator for a continuous nonlinear system cascaded with a data-hold device with minimum number of multipliers and linear discrete systems may be achieved by simplifying, carefully and systematically, the M.D.Z.T of the cascade combination of the data-hold device and the nonlinear system. However, it is observed that the simulator shown in Fig.5.10(b) is the optimum structure for the digital simulation of the continuous system shown in Fig.5.8(b).

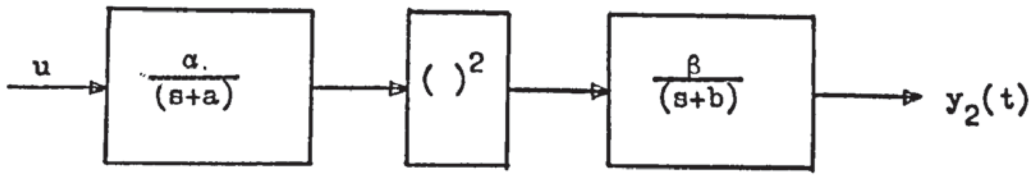


Fig.5.8(a) Continuous nonlinear system with a second-order kernel.

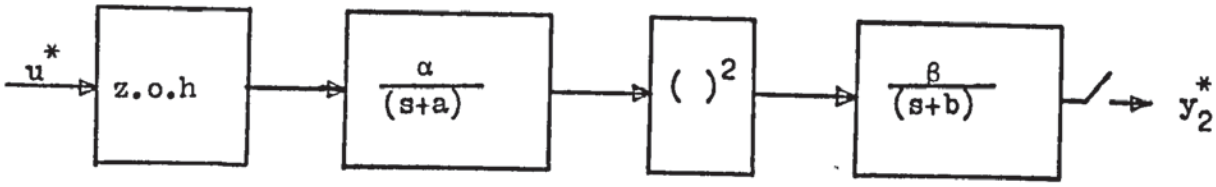


Fig.5.8(b) System for digital simulation.

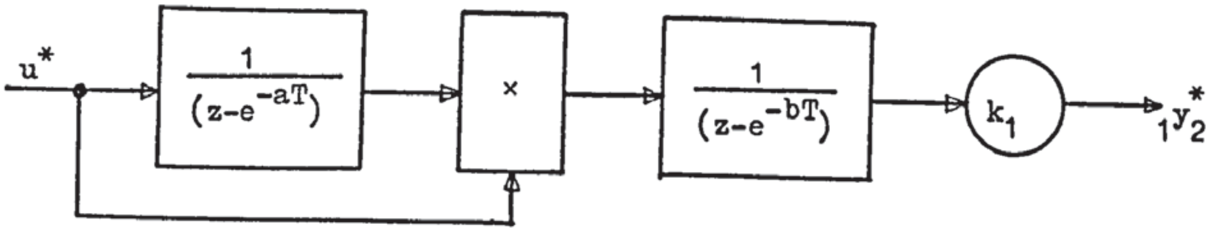


Fig.5.9(a) Synthesis of ${}_1P_2(z_1,z_2)$.

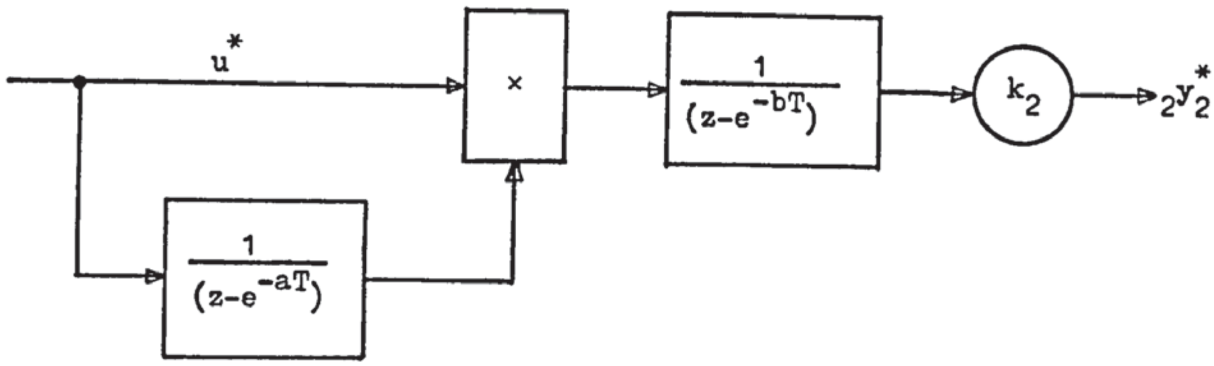


Fig.5.9(b) Synthesis of ${}_2P_2(z_1,z_2)$.

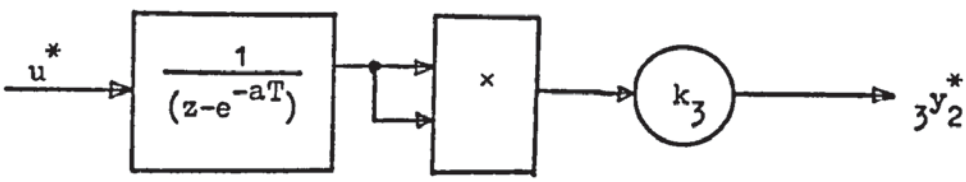


Fig.5.9(c) Synthesis of ${}_3P_2(z_1,z_2)$.

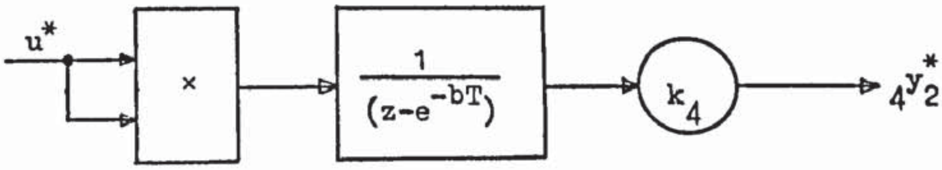


Fig.5.9(d) Synthesis of ${}_4P_2(z_1,z_2)$.

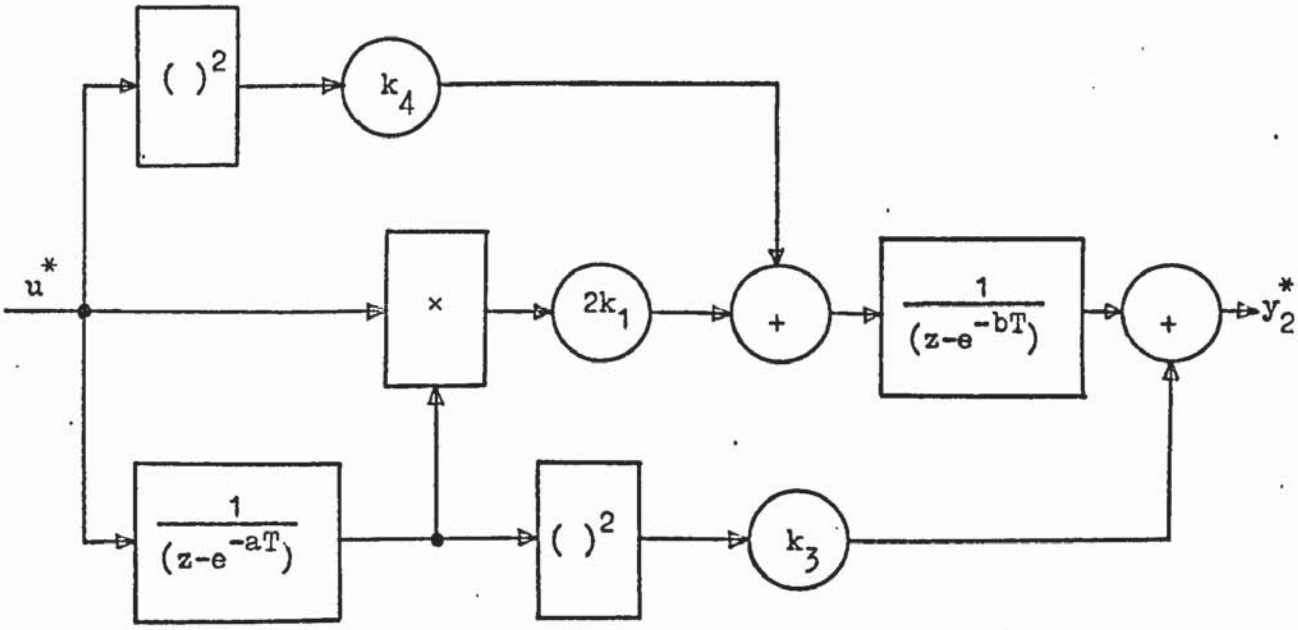


Fig.5.10(a) Discrete simulator for the system shown in Fig.5.8(b).

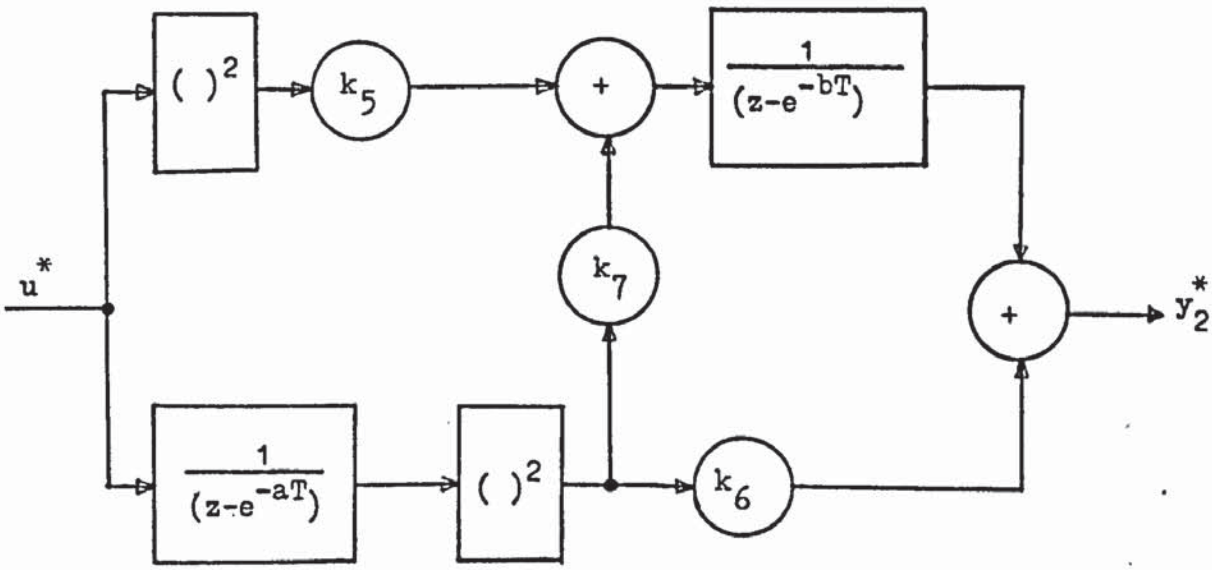


Fig.5.10(b) Alternative simulator for the system shown in Fig.5.8(b).

5.4.3 Example 2 - Simulation of Nonlinear System with 3^{rd}-Order Kernel

Consider now the continuous nonlinear system with a third-order kernel which is to be simulated as shown in Fig.5.11. The M.D.Z.T of the cascade combination may be obtined using the sequential process developed in Chapter 2 and may be written in the form

$$L_3(z_1,z_2,z_3) = k \sum_{i=1}^{5} {}_iP_3(z_1,z_2,z_3) , \tag{5.4.4}$$

where $${}_1P_3(z_1,z_2,z_3) = \frac{k_1}{\prod_{p=1}^{3} (z_p - e^{-aT})} , \quad {}_2P_3(z_1,z_2,z_3) = \frac{k_2}{(z_1z_2z_3 - e^{-bT})}$$

$${}_3P_3(z_1,z_2,z_3) = \frac{k_3}{(z_1z_2z_3 - e^{-bT})}\left[\sum_{p=1}^{3} \frac{1}{(z_p - e^{-aT})}\right] \tag{5.4.5}$$

$${}_4P_3(z_1,z_2,z_3) = \frac{k_4}{(z_1z_2z_3 - e^{-bT})}\left[\frac{1}{(z_1-e^{-aT})(z_2-e^{-aT})} + \frac{1}{(z_2-e^{-aT})(z_3-e^{-aT})} + \frac{1}{(z_1-e^{-aT})(z_3-e^{-aT})}\right]$$

$${}_5P_3(z_1,z_2,z_3) = \frac{k_5}{(z_1z_2z_3-e^{-bT})\prod_{p=1}^{3} (z_p-e^{-aT})} \quad \text{and } k = \frac{\alpha^3\beta}{(b-a)(b-2a)(b-3a)}$$

$$\begin{aligned}
k_1 &= b(b-a)(b-3a)(1-e^{-aT})^3 , \\
k_2 &= \left[abe^{-aT}\{3(b-3a)-(b-a)e^{-2aT}\} + 2a\{3a^2e^{-bT}-(b-a)(b-3a)\}\right], \\
k_3 &= ab(1- e^{-aT})\{e^{-3aT}(b-a)- e^{-aT}(b-3a)- 2ae^{-bT}\} , \\
k_4 &= ab(b-a)(e^{-bT}- e^{-3aT})(1- e^{-aT})^2 \\
k_5 &= - ab(b-a)(e^{-bT}- e^{-3aT})(1- e^{-aT})^3
\end{aligned} \tag{5.4.6}$$

Each of the kernels ${}_1P_3(z_1,z_2,z_3)$ to ${}_5P_3(z_1,z_2,z_3)$ can be synthesised using the procedure outlined in Appendix A.5 for third-order kernels and hence the complete discrete simulator for the system shown in Fig.5.11 may be realised as shown in Fig.5.12(a). This simulator requires six multipliers and two linear discrete systems. However, an attempt is made to reduce the number of multipliers by simplifying $L_3(z_1,z_2,z_3)$, further, but this is achieved at the complexity of the coefficients k_i. This may be observed from the alternative structure shown in Fig.5.12(b), for which $L_3(z_1,z_2,z_3)$ is given by

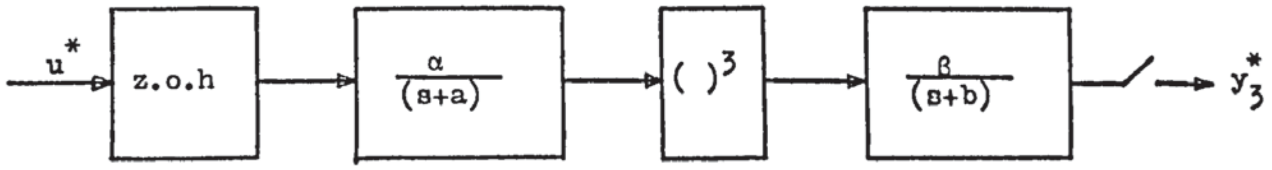


Fig.5.11 Simulation of continuous nonlinear system with 3^{rd}-order kernel.

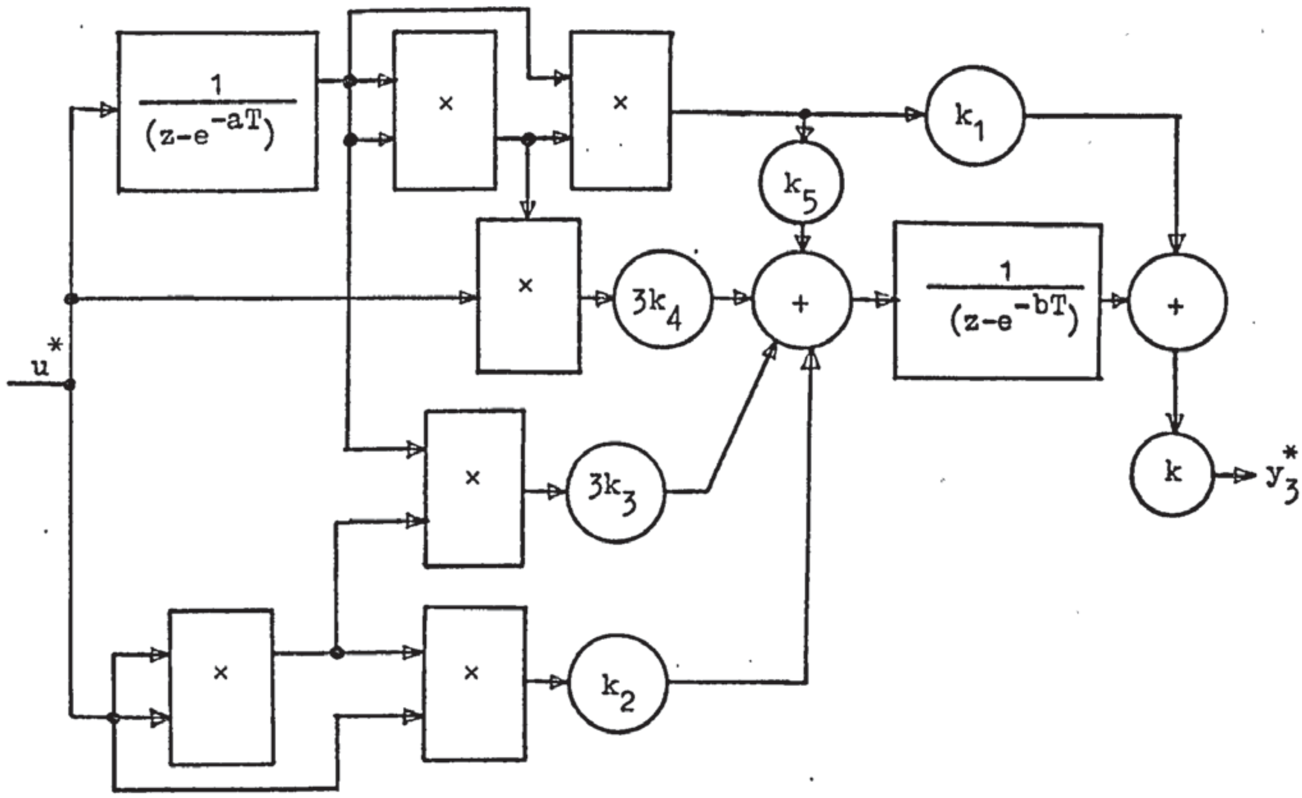


Fig.5.12(a) Discrete simulator for system shown in Fig.5.11.

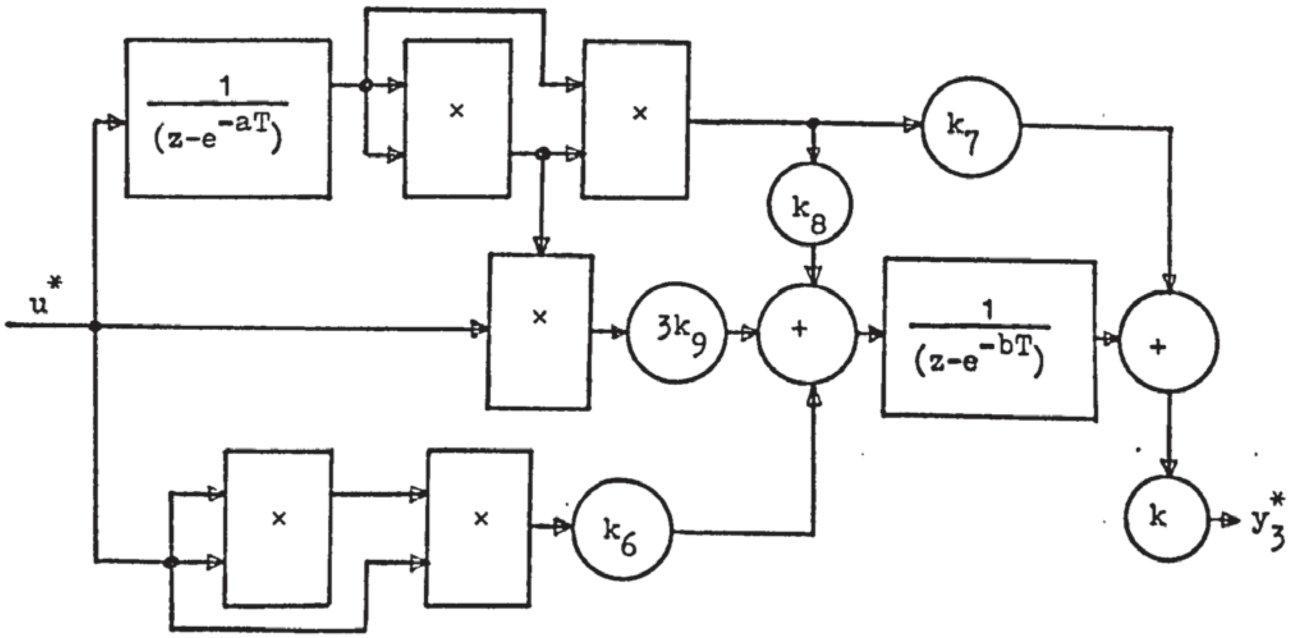


Fig.5.12(b) An alternative simulator for the system shown in Fig.5.11.

$$L_3(z_1,z_2,z_3) = k\left[\frac{k_6}{(z_1z_2z_3-e^{-bT})} + \frac{k_7}{\prod_{p=1}^{3}(z_p-e^{-aT})} + \frac{k_8}{(z_1z_2z_3-e^{-bT})\prod_{p=1}^{3}(z_p-e^{-aT})}\right.$$
$$+ \frac{k_9}{(z_1z_2z_3-e^{-bT})}\left\{\frac{1}{(z_1-e^{-aT})(z_2-e^{-aT})} + \frac{1}{(z_2-e^{-aT})(z_3-e^{-aT})}\right.$$
$$\left.\left. + \frac{1}{(z_1-e^{-aT})(z_3-e^{-aT})}\right\}\right] \qquad (5.4.7)$$

where

$$k_6 = \left[ab(1-e^{-aT})\{2a(1-e^{-bT})-(b-a)(1-e^{-aT})\}+ 2a^2\{b(1-e^{-aT})- 3a(1-e^{-bT})\}\right],$$
$$k_7 = b(1-e^{-aT})\left[(b-a)(1-e^{-aT})\{(b-2a)(1-e^{-aT})- 2a\}+ 2a^2(1-e^{-(b-a)T})\right],$$
$$k_8 = 2ab(1-e^{-aT})(e^{-bT}-e^{-3aT})\left[a(1-e^{-(b-a)T})-(b-a)(1-e^{-aT})\right], \text{ and}$$
$$k_9 = ab(1-e^{-aT})\left[(b-a)(1-e^{-aT})(e^{-bT}+e^{-2aT})+ 2ae^{-aT}(e^{-bT}-e^{-aT})\right] \qquad (5.4.8)$$

This discrete simulator, for the continuous system shown in Fig.5.11, requires only five multipliers and two linear systems. The validity of the simulator is demonstrsted by comparing the step response of the simulator with the step response of the system calculated from the formula

$$y(iT) = k\left[(b-a)(b-2a)(b-3a)- 3b(b-2a)(b-3a)e^{-aiT}+ 3b(b-a)(b-3a)e^{-2aiT}\right.$$
$$\left.- b(b-a)(b-2a)e^{-3aiT}+ 6a^3e^{-biT}\right] \qquad (5.4.9)$$

for T=1, $a=\frac{1}{3}$, b= 5/3. Eqn.(5.4.9) is the step response, obtained analytically, for the system shown in Fig.5.11. The step response of the system obtained from the simulator is shown in Fig.5.13, which agrees exactly with the calculated result.

5.4.4 Simulation of Volterra Kernels Characterising Direction Dependent System

This system is a first-order system whose dynamics is dependent on the direction of the input signal[89,92]. The system to be simulated is described by the equations

$$T_u\,\dot{y} + y = u \qquad \dot{y} \text{ positive} \qquad (5.4.10)$$
$$T_D\,\dot{y} + y = u \qquad \dot{y} \text{ negative} \qquad (5.4.11)$$

where T_u and T_D are time constants of the system. Since the input signal

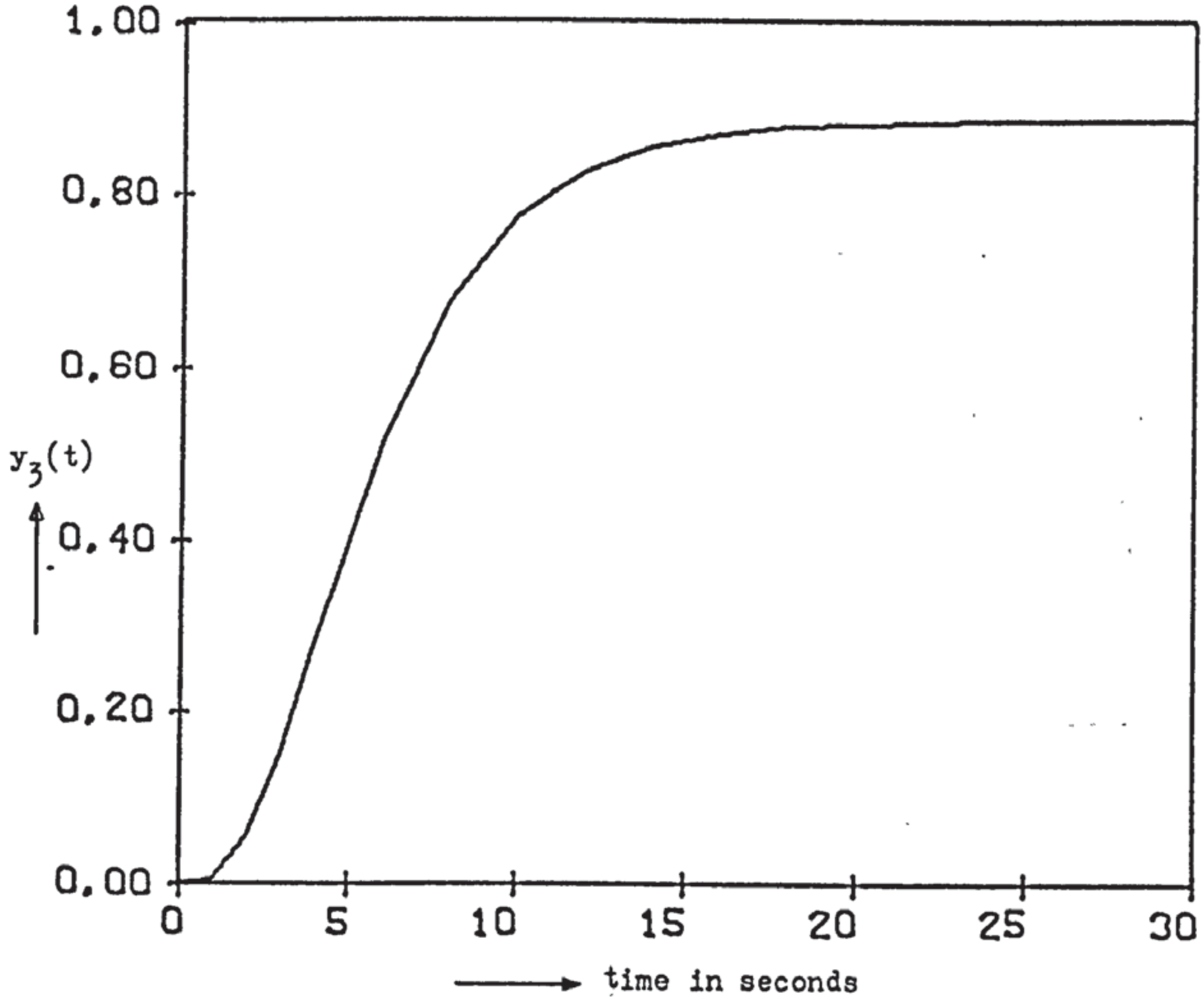


Fig.5.13 Step response of system shown in Fig.5.11.

is a positive or a negative step, eqns.(5.4.10) and (5.4.11) may be combined to give

$$\frac{\dot{y}}{(r\,u + \omega)} + y = u \tag{5.4.12}$$

where $r = \frac{1}{2}(\frac{1}{T_u} - \frac{1}{T_D})$ and $\omega = \frac{1}{2}(\frac{1}{T_u} + \frac{1}{T_D})$. Eqn.(5.4.12) represents a nonlinear feedback system, shown in Fig.5.14, which may be characterised by a finite number of Volterra kernels. The M.D.L.T of the first four Volterra kernels are given by

$$W_1(s) = \frac{\omega}{s+\omega} \quad , \qquad W_2(s_1,s_2) = \frac{r\,s_2}{(s_1+s_2+\omega)(s_2+\omega)}$$

$$W_3(s_1,s_2,s_3) = \frac{-r^2 s_3}{(s_1+s_2+s_3+\omega)(s_2+s_3+\omega)(s_3+\omega)} \quad \text{and} \tag{5.4.13}$$

$$W_4(s_1,s_2,s_3,s_4) = \frac{r^3 s_4}{(s_1+s_2+s_3+s_4+\omega)(s_2+s_3+s_4+\omega)(s_3+s_4+\omega)(s_4+\omega)}$$

Since the response of the system shown in Fig.5.14 is equal to the sum of the responses of all the kernels, the system can be simulated by simulating all the kernels. The kernels $W_1(s)$ to $W_4(s_1,s_2,s_3,s_4)$, to be simulated, are shown in Figs.5.15(a) to (d), respectively. The M.D.Z.T of the kernels associated with a zero-order hold are given by

$$L_1(z) = \frac{(1-e^{-\omega T})}{(z-e^{-\omega T})} \quad , \qquad L_2(z_1,z_2) = \frac{rTe^{-\omega T}(z_2-1)}{(z_1z_2-e^{-\omega T})(z_2-e^{-\omega T})}$$

$$L_3(z_1,z_2,z_3) = \frac{-r^2T^2e^{-\omega T}(z_3-1)(z_2z_3+e^{-\omega T})}{2(z_1z_2z_3-e^{-\omega T})(z_2z_3-e^{-\omega T})(z_3-e^{-\omega T})} \quad \text{and} \tag{5.4.14}$$

$$L_4(z_1,z_2,z_3,z_4) = \frac{r^3T^3e^{-\omega T}(z_4-1)\left[z_2z_3z_4(z_3z_4+2e^{-\omega T})+2e^{-\omega T}(z_3z_4+\frac{1}{2}e^{-\omega T})\right]}{6(z_1z_2z_3z_4-e^{-\omega T})(z_2z_3z_4-e^{-\omega T})(z_3z_4-e^{-\omega T})(z_4-e^{-\omega T})}$$

The first-order simulator represented by $L_1(z)$ can be easily realised as shown in Fig.5.16(a). The realisation of higher-order simulators is now described.

(a) <u>Second-order Simulator:</u> The second-order z transform kernel $L_2(z_1,z_2)$ is in the form, for m=0,

$$L_2(z_1,z_2) = {}_1P_2(z_1,z_2) = k_2\;{}_3J_{11}(z_1z_2)\,{}_2J_{11}(z_2)\,{}_1J_{11}(z_1) \tag{5.4.15}$$

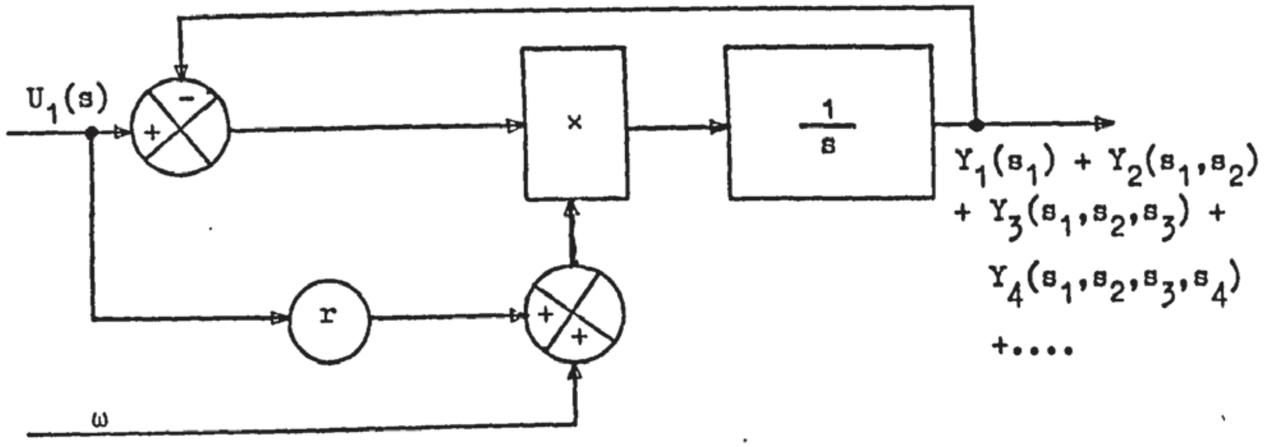


Fig.5.14 Block diagram of a system with direction dependent dynamic responses.

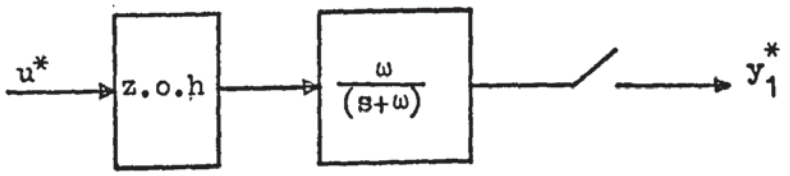


Fig.5.15(a) First-order kernel for simulation.

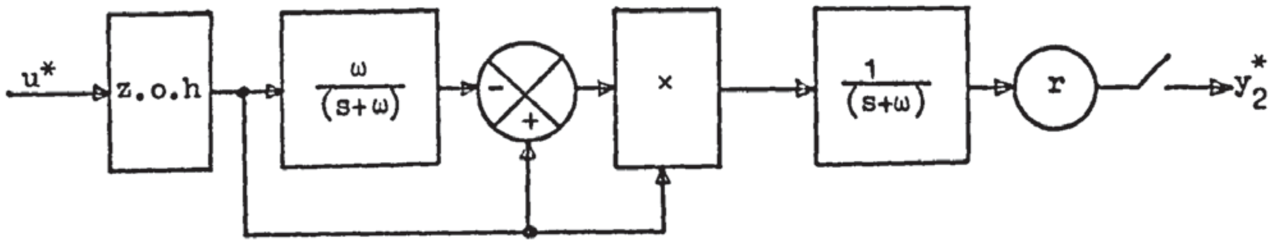


Fig.5.15(b) Second-order kernel for simulation.

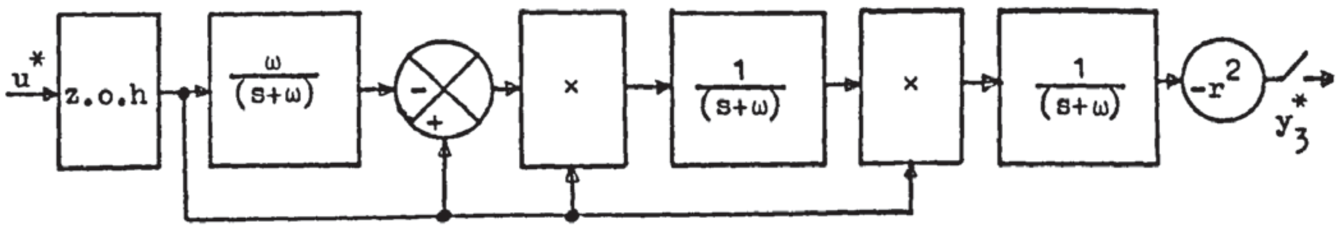


Fig.5.15(c) Third-order kernel for simulation.

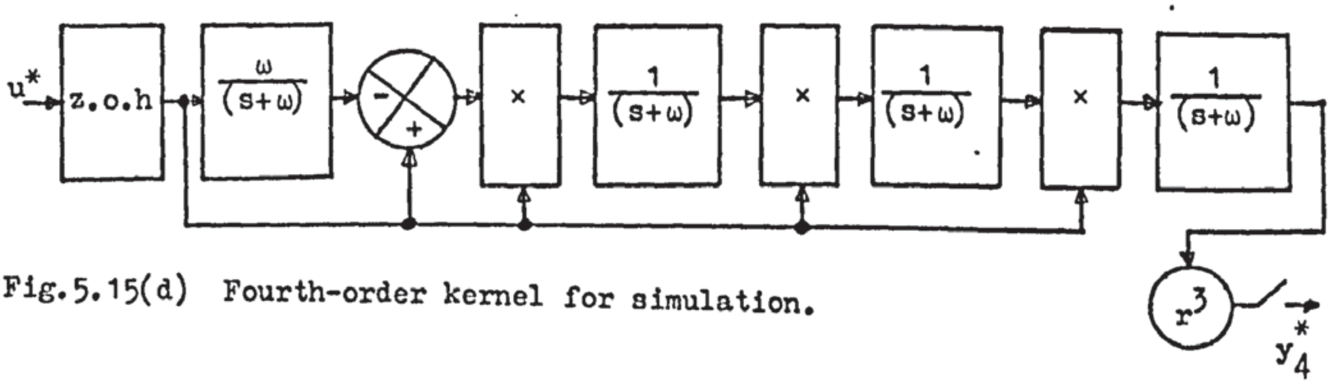


Fig.5.15(d) Fourth-order kernel for simulation.

and hence can be synthesised, using the procedure described in Table 5.1, as shown in Fig.5.16(b). This simulator requires only one multiplier and two linear discrete systems, and hence is a typical second-order canonic form shown in Fig.5.1.

(b) Third-order Simulator: The third-order kernel $L_3(z_1,z_2,z_3)$ is in the form, for m=0,

$$L_3(z_1,z_2,z_3) = {}_1P_3(z_1,z_2,z_3) = k_3\ {}_5J_{11}(z_1z_2z_3)\,{}_4J_{11}(z_3)\,{}_3J_{11}(z_1z_2) \times {}_2J_{11}(z_2)\,{}_1J_{11}(z_1) \qquad (5.4.16)$$

and hence can be realised, using the procedure described in Appendix A.5, as shown in Fig.5.16(c). This third-order simulator, which requires only two multipliers and three linear systems, represents typically third-order canonic form shown in Fig.5.3(a).

(c) Fourth-order Simulator: The fourth-order kernel $L_4(z_1,z_2,z_3,z_4)$ is expressible in the form, for m=0, as

$$L_4(z_1,z_2,z_3,z_4) = \sum_{i=1}^{2} {}_iP_{41}(z_1,z_2,z_3,z_4)$$

$$= \sum_{i=1}^{2} k_{4i}\ {}_7J_{1i}(z_1z_2z_3z_4)\,{}_6J_{1i}(z_4)\,{}_5J_{1i}(z_1z_2z_3)\,{}_4J_{1i}(z_3) \times {}_3J_{1i}(z_1z_2)\,{}_2J_{1i}(z_2)\,{}_1J_{1i}(z_1) \qquad (5.4.17)$$

Each of the kernels ${}_iP_{41}(z_1,z_2,z_3,z_4)$, for i =1,2 can be synthesised in the first fourth-order canonic form shown in Fig.5.4(a), and the complete structure for the fourth-order simulator is shown in Fig.5.16(d). This simulator requires four multipliers and six linear discrete systems.

Hence, the response of the direction dependent system shown in Fig. 5.14 may be obtained by adding the outputs of all the simulators shown in Figs.5.16(a) to (d), which represent the linear, quadratic, cubic and bi-quadratic components, respectively. The validity of the simulator is demonstrated by comparing the total output obtained by adding the outputs of all the simulators, with the actual response and the results are in good agreement. The actual response of the system is given by

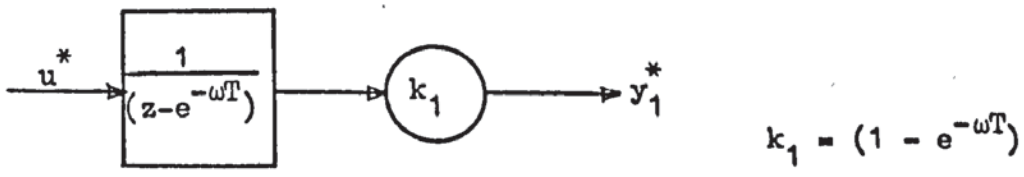


Fig.5.16(a) First-order simulator, $L_1(z)$.

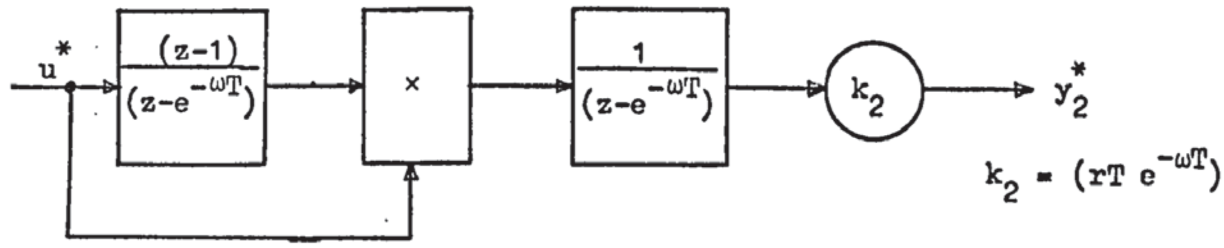


Fig.5.16(b) Second-order simulator, $L_2(z_1,z_2)$.

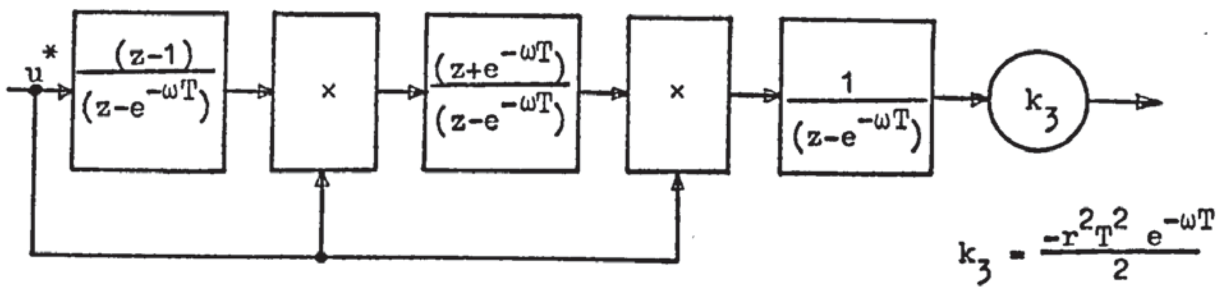


Fig.5.16(c) Third-order simulator, $L_3(z_1,z_2,z_3)$.

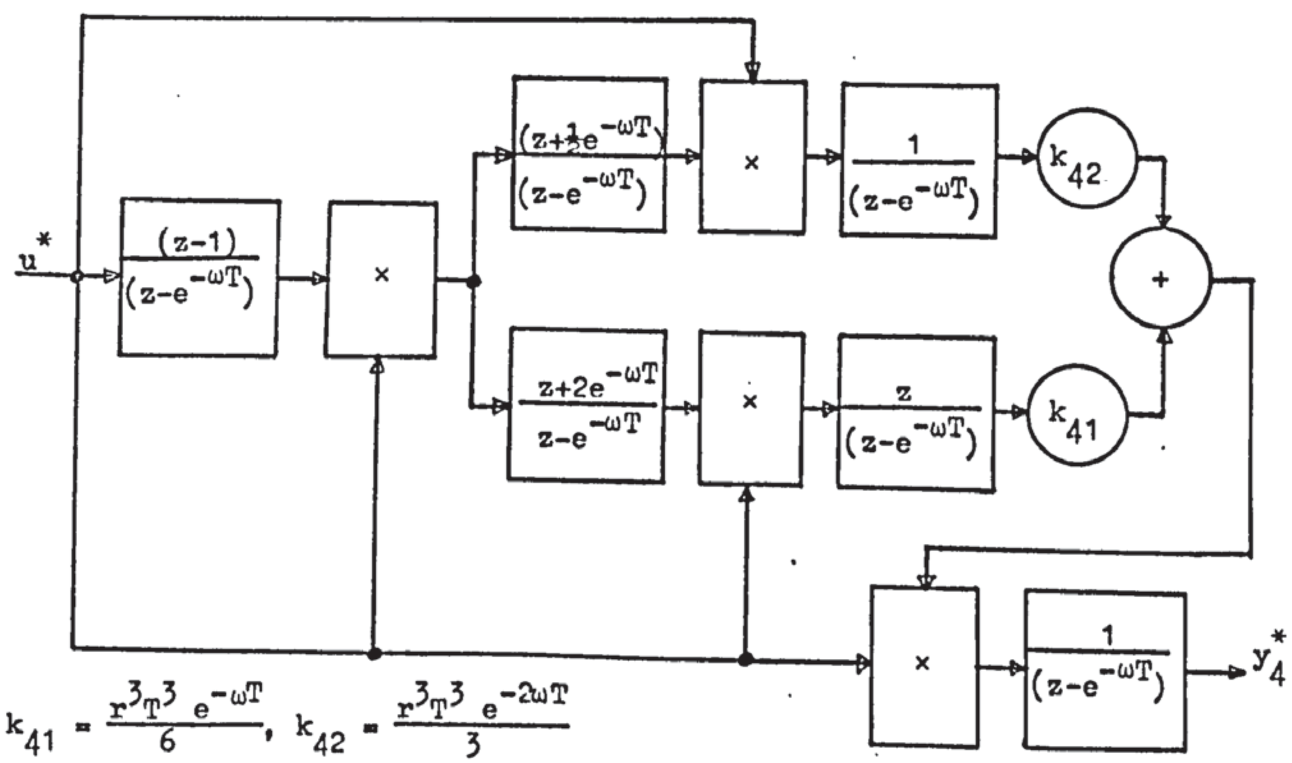


Fig.5.16(d) Fourth-order simulator, $L_4(z_1,z_2,z_3,z_4)$.

$$\left[y(iT)\right]_+ = (1- e^{-(\omega+r)iT}) \text{ , for a positive step input, and by}$$
$$\left[y(iT)\right]_- = (-1+ e^{-(\omega-r)iT}) \text{ , for a negative step input.} \qquad (5.4.18)$$

The actual response, as obtained from the above equations and the total response of the simulated system are shown in Fig.5.17, for $T=1$, $r=0.125$ and $\omega=0.375$. It may be observed, from the figure, that the Volterra series solution converges rapidly to the actual response.

5.5 Conclusions

A simple and systematic procedure has been described here for the synthesis of a large class of nonlinear discrete systems. The oscillating type kernels, which are frequently encountered in nonlinear feedback discrete systems, can be realised using a finite number of multipliers and second-order linear discrete systems whereas the exponentially decaying type kernels can be synthesised using a finite number of multipliers and first-order linear discrete systems.

The synthesis procedure developed here may also be used for obtaining a discrete simulator for a continuous nonlinear system cascaded with a data-hold device. Three illustrative examples have been given: the first two examples clearly show how a discrete simulator with a minimum number of multipliers can be obtained for the digital simulation of continuous nonlinear systems, while the third example was chosen to demonstrate the digital simulation of continuous nonlinear feedback systems.

A limitation of the synthesis method described here is that the algebraic computation increases enormously when Volterra kernels of order four and higher are to be synthesised. However, this method is preferable for systems having small nonlinearities.

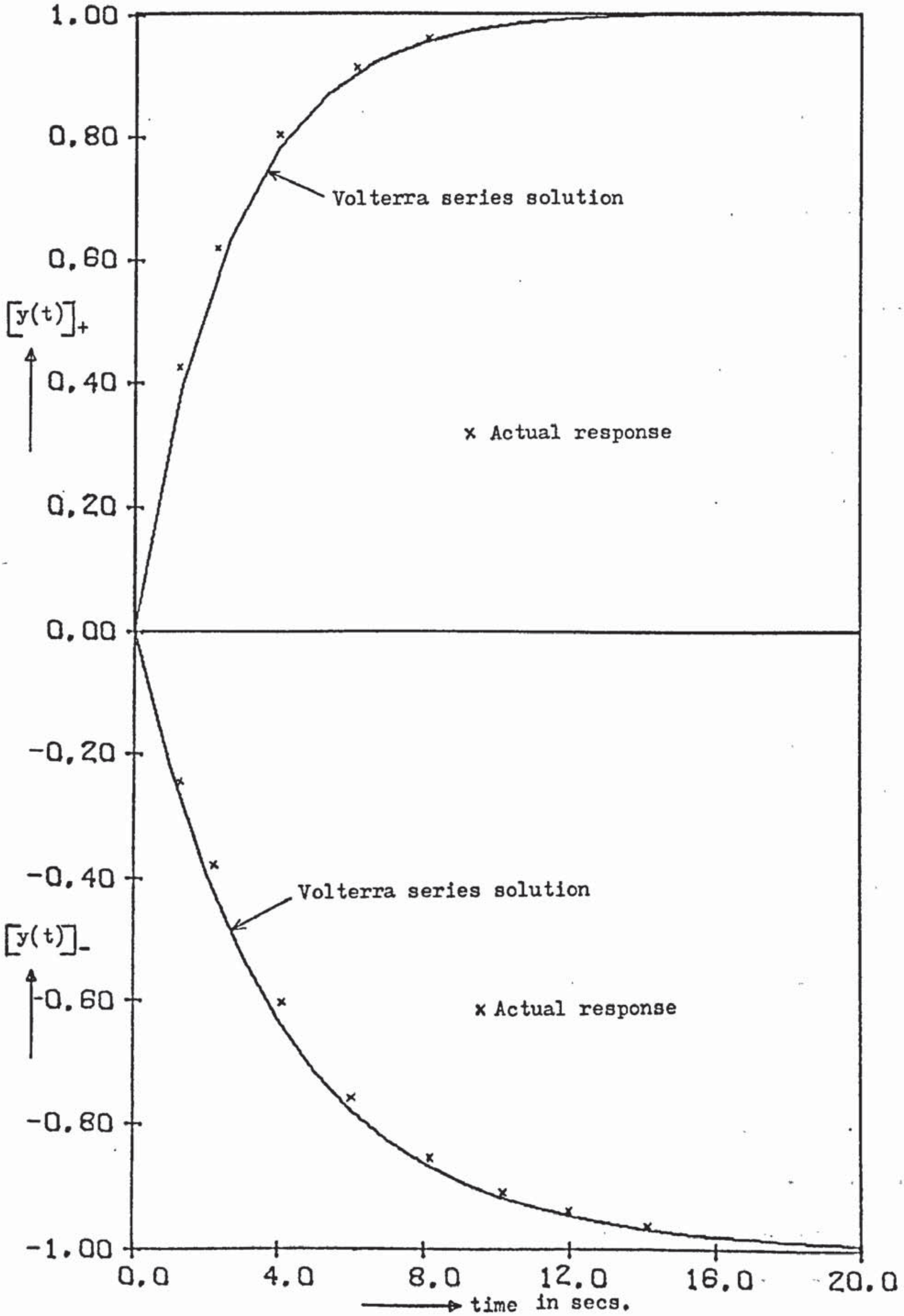


Fig.5.17 Response of the direction dependent system, for positive and negative step inputs.

CHAPTER 6

STATE VARIABLE DESCRIPTION OF NONLINEAR SYSTEMS-

CONTINUOUS SYSTEMS

6.1 Introduction

The advantage of the state variable method over the conventional transfer function method is that the state variable approach allows a unified representation of continuous and digital systems, single variable and multivariable systems. The state variable formulation is also convenient for computer solutions. Although the use of transform methods in the state space solution of linear control systems[83] is well established, no general attempt has so far been made for the state variable solution of Volterra nonlinear systems using multidimensional transform methods. Ferguson[93] has characterised single-input, single-output nonlinear systems by a power-series state differential equation and presented algorithms to construct the equivalent difference equation for computer implementation, but the solution was given in time domain only. Alper[24] has given a state variable solution of a nonlinear differential equation in transforms, which by no means is complete and general. The advantage of formulation and solution of the state variable problem in the transform domain is that it allows an explicit input-output relationship to be established in terms of the state transition matrix of the system, which then leads to the derivation and synthesis of multidimensional transform kernels of the system in terms of the linear state transition matrix.

This chapter presents a method of representing nonlinear continuous systems by power series of Volterra form using multi-dimensional arrays. The state equation of a continuous system is written as a vector, in which each component is taken as a power-series combination of the components of a state vector. The output equation of the system is similarly expanded in terms of the components of the state vector. The dynamic equations(i.e., state and output equations) are then generalised to include the zero-order component in the signals. Employing the Volterra series and the multidimensional Laplace transforms, a complete solution of the dynamic equations is obtained in terms of linear state transition matrix. To illustrate the method, the solution is used to obtain the Volterra series expansion of the outputs of a feedback nonlinear system(direction dependent system) and an open-loop linear-square-linear system. The formulation and the solution of the dynamic equations are then extended to the case of multivariable nonlinear systems. The relationship between the state transition matrix and the Volterra kernels of a multivariable system is established, by which the multidimensional Laplace transform kernels may be derived and synthesised in terms of the linear state transition matrix. The method is illustrated by analysing the response of a two-input, single-output diode ring multiplier circuit.

6.2 Representation of Continuous Nonlinear Systems

This section describes how the continuous nonlinear system with multiplicative intercoupling of states and the states with the input, may be characterised by a set of dynamic equations. The dynamic equations are then modified to take into account the zero-order component in the signals which is constant for all time. The solution of the dynamic equations is then obtained, in the form of generalised Volterra series expansion, using multidimensional Laplace transforms. Two examples are solved to illustrate the method of solution as well as to show the effect of the zero-order component in the input signal, on the response of the system.

6.2.1 State and Output Equations of a Single-Input, Single-Output System

The state equation of a most general multiplicative[93] nonlinear time-invariant system may be written, using Einstein summation convention[94] as

$$ {}_p\dot{v}(t) = {}_pA_i\ {}_iv(t) + {}_pB_{ij}\ {}_iv(t)\ {}_jv(t) + {}_pC_{ijk}\ {}_iv(t)\ {}_jv(t)\ {}_kv(t) + \ldots. \quad (6.2.1) $$

where the repeated indices imply summation to N, the left and right hand subscripts differentiate between the symmetric and nonsymmetric elements of multidimensional arrays A, B and C. The above equation defines a vector ${}_p\dot{v}(t)$, in which each component is a power-series combination of the components of an augmented state vector ${}_pv(t)$ defined by a (P+1)x1 column vector,

$$ {}_pv(t) = \mathrm{Col}\ \{{}_1x(t)\ {}_2x(t)\ {}_3x(t) \ldots {}_Px(t)\ \ u(t)\} \quad (6.2.2) $$

where P is the number of state variables, ${}_ix$ and ${}_jx$ are state variables of the system represented by outputs of i^{th} and j^{th} integrators, respectively, in which the nonlinearities are in the form of multiplicative intercoupling of states and the states with the input.

The output equation may be similarly written in the form, as

$$ y(t) = E_i\ {}_iv(t) + F_{ij}\ {}_iv(t)\ {}_jv(t) + G_{ijk}\ {}_iv(t)\ {}_jv(t)\ {}_kv(t) + \ldots. \quad (6.2.3) $$

where E, F, G are multidimensional arrays. The analogue model of the system, having multiplicative intercoupling of states and the states with the input, is shown in Fig.6.1.

6.2.2 Dynamic Equations for Systems with Zero-Order Component in Signals

In the previous subsection, the signals in the system were assumed to have no zero-order component. However, in certain industrial plants, the signals in the system(including the input and the output) may consist of a constant level besides the time dependent function. Here, the dynamic equations are generalised to characterise such nonlinear systems. The dynamic equations characterising the system with input $u(t) = u_0 + u_1(t)$ are

$$ {}_p\dot{x} = {}_pA'_i\ {}_ix + {}_pD'u + {}_pB'_{ij}\ {}_iv'\ {}_jv' + {}_pC'_{ijk}\ {}_iv'\ {}_jv'\ {}_kv' + \ldots. \quad (6.2.4) $$

$$ y = E'_i\ {}_iv' + F'_{ij}\ {}_iv'\ {}_jv' + G'_{ijk}\ {}_iv'\ {}_jv'\ {}_kv' + \ldots \quad (6.2.5) $$

where ${}_ix$ is a Px1 state vector given by ${}_ix = \mathrm{col}\{{}_1x\ {}_2x \ldots {}_Px\}$, ${}_iv'$ is

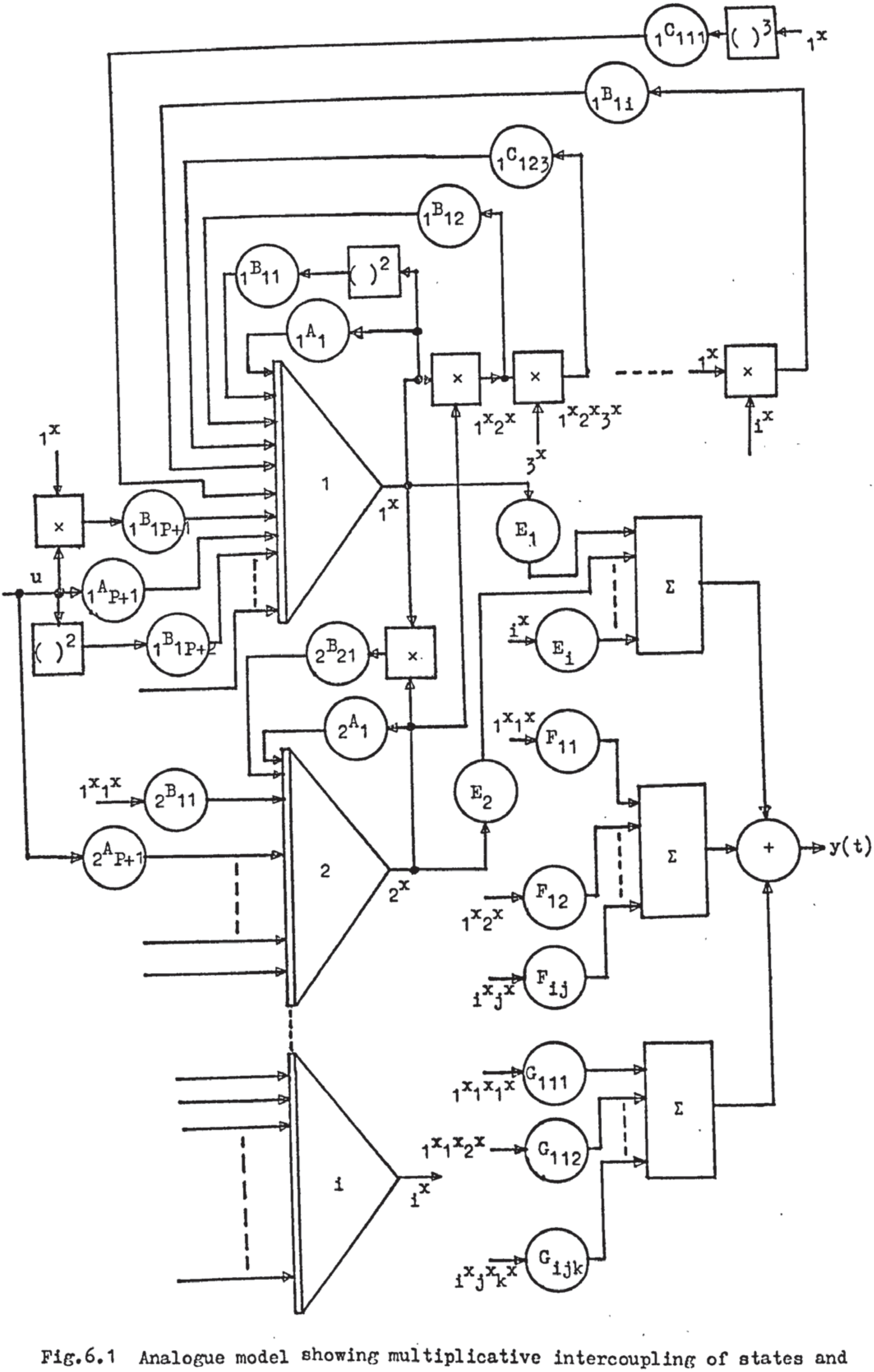


Fig.6.1 Analogue model showing multiplicative intercoupling of states and states with the input.

a (P+1)x1 augmented state vector to include u, A', B', C', E', F' and G' are the augmented forms of A, B, C, E, F and G respectively, to include the coupling of u and states ${}_ix$, and u_0 is the zero-order component in the input signal. In transforms, the input is represented by $U(s) = U_0 + U_1(s)$ where $U_0 = u_0$. It should be noted that the first two terms of eqn.(6.2.4) characterise the linear behaviour of the system and the rest of the terms characterise the nonlinearities in the system.

6.2.3 Method of Solution

Since eqns.(6.2.4) and (6.2.5) are nonlinear equations, their solution is sought in the form of generalised Volterra series expansion. To obtain the solution of the state equation in terms of the initial condition vector ${}_ix(0)$ and the input variable u(t), eqn.(6.2.4) is Laplace transformed to give

$$\{sI - A'\}\,{}_pX(s) = {}_px(0) + {}_pD'U(s) + {}_pB'_{ij}\,{}_iV'(s) * {}_jV'(s) + {}_pC'_{ijk}\,{}_iV'(s) * {}_jV'(s) * {}_kV'(s) + \ldots \quad (6.2.6)$$

where the initial condition vector ${}_px(0)$ should not be confused with the zero-order component vector ${}_px_0$. Pre-multiplying either side by $\Phi'(s) = [sI - A']^{-1}$, gives

$${}_pX(s) = {}_p\Phi'_i(s)\Big[{}_ix(0) + {}_iD'U(s) + {}_iB'_{jk}\,{}_jV'(s) * {}_kV'(s) + {}_iC'_{jkl}\,{}_jV'(s) * {}_kV'(s) * {}_lV'(s) + \ldots\Big] \quad (6.2.7)$$

The signal ${}_ix(t)$ representing the state of the nonlinear system, is assumed to be in the generalised Volterra series form as

$${}_ix = {}_ix_0 + {}_ix_1(t_1) + {}_ix_2(t_1, t_2) + {}_ix_3(t_1, t_2, t_3) + \ldots \quad (6.2.8)$$

The transform of the above equation is given by

$${}_iX = {}_iX_0 + {}_iX_1(s) + {}_iX_2(s_1, s_2) + {}_iX_3(s_1, s_2, s_3) + \ldots + {}_iX_n(s_1, s_2, \ldots, s_n) \quad (6.2.9a)$$

where ${}_iX_0 = {}_ix_0$. Similarly, the transform of ${}_iv'(t)$ is given by

$${}_iV' = {}_iV'_0 + {}_iV'_1(s) + {}_iV'_2(s_1, s_2) + {}_iV'_3(s_1, s_2, s_3) + \ldots + {}_iV'_n(s_1, s_2, \ldots, s_n) \quad (6.2.9b)$$

where ${}_iV_0' = {}_iv_0'$. Substituting for ${}_pX(s)$ and ${}_pV'(s)$ in eqn.(6.2.7) from the above equations, equating terms of equal order on either side and using the linear system operation of Chapter 4, yields the terms of the generalised Volterra series solution as

$$ {}_pX_0 = {}_p\Phi_i'(0)\left[{}_iD'U_0 + {}_iB_{jk}'\,{}_jV_0'\,{}_kV_0' + {}_iC_{jkl}'\,{}_jV_0'\,{}_kV_0'\,{}_lV_0'\right] \tag{6.2.10} $$

$$ \begin{aligned} {}_pX_1(s) = {}_p\Phi_i'(s)\Big[&{}_ix(0) + {}_iD'U_1(s) + {}_iB_{jk}'\{{}_jV_0'\,{}_kV_1'(s) + {}_jV_1'(s)\,{}_kV_0'\} \\ &+ {}_iC_{jkl}'\{{}_jV_0'\,{}_kV_1'(s)\,{}_lV_0' + {}_jV_0'\,{}_kV_0'\,{}_lV_1'(s) + {}_jV_1'(s)\,{}_kV_0'\,{}_lV_0'\}\Big] \end{aligned} \tag{6.2.11} $$

and the n^{th} order term of the series, for $n > 1$, is given by

$$ \begin{aligned} {}_pX_n(s_1,s_2,\ldots,s_n) = {}_p\Phi_i'\Big(\sum_{p=1}^{n} s_p\Big) \\ \Big[&{}_iB_{jk}'\{{}_jV_0'\,{}_kV_n'(s_1,\ldots,s_n) + {}_jV_n'(s_1,\ldots,s_n)\,{}_kV_0' \\ &\quad + \sum_{m=1}^{n-1} {}_jV_m'(s_1,\ldots,s_m)\,{}_kV_{n-m}'(s_{m+1},\ldots,s_n)\} \\ &+ {}_iC_{jkl}'\{{}_jV_0'\,{}_kV_0'\,{}_lV_n'(s_1,\ldots,s_n) + {}_jV_0'\,{}_kV_n'(s_1,\ldots,s_n)\,{}_lV_0' \\ &+ {}_jV_n'(s_1,\ldots,s_n)\,{}_kV_0'\,{}_lV_0' \\ &+ {}_jV_0'\sum_{m=1}^{n-1} {}_kV_m'(s_1\ldots,s_m)\,{}_lV_{n-m}'(s_{m+1},\ldots,s_n) \\ &+ {}_kV_0'\sum_{m=1}^{n-1} {}_jV_m'(s_1,\ldots,s_m)\,{}_lV_{n-m}'(s_{m+1},\ldots,s_n) \\ &+ {}_lV_0'\sum_{m=1}^{n-1} {}_jV_m'(s_1,\ldots,s_m)\,{}_kV_{n-m}'(s_{m+1},\ldots,s_n)\} \\ &+ {}_iC_{jkl}'\{\sum_{m=1}^{n-2} {}_jV_m'(s_1,\ldots,s_m)\sum_{q=1}^{n-m-1} {}_kV_q'(s_{m+1},\ldots,s_{m+q}) \\ &\qquad {}_lV_{n-m-q}'(s_{m+q+1},\ldots,s_n)\}\Big] \end{aligned} $$

$$ n > 1 \tag{6.2.12} $$

where $\phi'(t) = L^{-1}\{\Phi'(s)\}$ is the linear state transition matrix, which generates all the components of the state vector.

Assuming that the series in eqn.(6.2.9a) converges for some value of n, the eqns.(6.2.10) to (6.2.12) provide a complete solution to the state equation. Equation(6.2.11),which represents the solution of the linear part of eqn.(6.2.1), shows, explicitly, that the effects of the zero-order component vector, the initial condition vector and the input are separate(superposition applies). On the other hand, eqn.(6.2.12), which provides the solution for the nonlinear part of the state equation, for $n > 1$, shows the interaction between the initial condition vector, the zero-order component vector, the state vector and the input. The state transition equation which describes the state of the system for $t > t_0$ is derived in the next chapter, where t_0 is the initial time.

To obtain the output, eqn.(6.2.5) is Laplace transformed which gives

$$Y(s) = E_i'\ {}_iV'(s) + F_{ij}'\ {}_iV'(s) *_j V'(s) + G_{ijk}'\ {}_iV'(s) *_j V'(s) *_k V'(s) + \ldots \tag{6.2.13}$$

The solution to the output equation may be obtained by substituting the generalised Volterra series expansions of $Y(s)$ and ${}_pV'(s)$ into eqn. (6.2.13) and equating the terms of equal order. The result is

$$Y_0 = \left[E_i'\ {}_iV_0' + F_{ij}'\ {}_iV_0'\ {}_jV_0' + G_{ijk}'\ {}_iV_0'\ {}_jV_0'\ {}_kV_0'\right] \tag{6.2.14}$$

$$\begin{aligned} Y_1(s) = \Big[E_i'\ {}_iV_1'(s) &+ F_{ij}'\{{}_iV_0'\ {}_jV_1'(s) + {}_iV_1'(s)\ {}_jV_0'\} + G_{ijk}'\{{}_iV_0'\ {}_jV_1'(s)\ {}_kV_0' \\ &+ {}_iV_0'\ {}_jV_0'\ {}_kV_1'(s) + {}_iV_1'(s)\ {}_jV_0'\ {}_kV_0'\}\Big] \end{aligned} \tag{6.2.15}$$

and

$$\begin{aligned} &Y_n(s_1, s_2, \ldots, s_n) \\ &= \Big[E_i'\ {}_iV_n'(s_1,\ldots,s_n) + F_{ij}'\{{}_iV_0'\ {}_jV_n'(s_1,\ldots,s_n) + {}_iV_n'(s_1,\ldots,s_n)\ {}_jV_0' \\ &\quad + \sum_{m=1}^{n-1} {}_iV_m'(s_1,\ldots,s_m)\ {}_jV_{n-m}'(s_{m+1},\ldots,s_n)\} + G_{ijk}'\{{}_iV_0'\ {}_jV_0'\ {}_kV_n'(s_1,\ldots,s_n) \\ &\quad + {}_iV_0'\ {}_jV_n'(s_1,\ldots,s_n)\ {}_kV_0' + {}_iV_0' \sum_{m=1}^{n-1} {}_jV_m'(s_1,\ldots,s_m)\ {}_kV_{n-m}'(s_{m+1},\ldots,s_n) \\ &\quad + {}_iV_n'(s_1,\ldots,s_n)\ {}_jV_0'\ {}_kV_0' + {}_jV_0' \sum_{m=1}^{n-1} {}_iV_m'(s_1,\ldots,s_m)\ {}_kV_{n-m}'(s_{m+1},\ldots,s_n) \\ &\quad + {}_kV_0' \sum_{m=1}^{n-1} {}_iV_m'(s_1,\ldots,s_m)\ {}_jV_{n-m}'(s_{m+1},\ldots,s_n)\} + G_{ijk}'\{\sum_{m=1}^{n-2} {}_iV_m'(s_1,\ldots,s_m) \\ &\quad \sum_{q=1}^{n-m-1} {}_jV_q'(s_{m+1},\ldots,s_{m+q})\ {}_kV_{n-m-q}'(s_{m+q+1},\ldots,s_n)\}\Big], \quad n > 1 \end{aligned} \tag{6.2.16}$$

where ${}_iV'_0$, ${}_iV'_1(s), \ldots, {}_iV'_n(s_1,s_2,\ldots,s_n)$ may be obtained from eqns. (6.2.10) to (6.2.12) and the input transform, since ${}_iV'_0=\left[{}_iX_0 \quad U_0\right]$, ${}_iV'_1(s)=\left[{}_iX_1(s) \quad U_1(s)\right]$ and ${}_iV'_n(s_1,\ldots,s_n)= {}_iX_n(s_1,\ldots,s_n)$, for $n>1$.

If the Volterra series expansion of the output is convergent for first few terms of the series, then eqns.(6.2.14) to (6.2.16) provide a complete solution to the output equation. Using the association of variables procedure, $Y_n(s_1,s_2,\ldots,s_n)$ may be reduced to one-dimensional transform $Y_1(s)$ which may then be inverted to give the system's response $y_n(t)$. It is to be noted that the zero-order component in the system output, which is constant for all time, is given by $y_0=Y_0$ where Y_0 is given by eqn.(6.2.14).

If the zero-order component U_0 in the input signal is zero, then ${}_pX_0= 0$ is a stable solution, for which $Y_0= 0$ and hence eqns.(6.2.11) and (6.2.15) become

$$ {}_pX_1(s)= {}_p\Phi_i(s)\ {}_ix(0)+ {}_p\Phi_i(s)\ {}_iD\ U_1(s) \tag{6.2.17}$$

$$ Y_1(s) = E^a_i\ {}_i\Phi_j(s)\ {}_jx(0)+\{E^a_i\ {}_i\Phi_j(s)\ {}_jD + E^b\}\ U_1(s) \tag{6.2.18}$$

in which the array E is partitioned between the state vector and the input. It can be seen at once that eqns.(6.2.17) and (6.2.18) represent the solution of the dynamic equations in linear system theory. Thus, ${}_pX_n(s_1,s_2,\ldots,s_n)$ and $Y_n(s_1,s_2,\ldots,s_n)$, for $n>1$, represent the solution of the nonlinear part of the dynamic eqns.(6.2.4) and (6.2.5), respectively.

6.2.4 Example - Direction Dependent System(Feedback Nonlinear System)

To illustrate the method, consider the direction dependent system shown in Fig.5.14 of chapter 5, characterised by the system equation

$$ \frac{dy}{dt} + \omega\ y + r\ u\cdot y = \omega\ u + r\ u\cdot u \tag{6.2.19}$$

where $u(t)= u_0+ u_1(t)$, is the input signal. The state variable is chosen as ${}_1x = y$.

The dynamic equations are written as

$${}_1\dot{x} = {}_1A_1'\ {}_1x + {}_1A_2'\ u + {}_1B_{11}'\ {}_1x.u + {}_1B_{21}'\ u.u \quad \text{and} \quad y = E_1'\ {}_1x \qquad (6.2.20)$$

where ${}_1A_1' = -\omega$, ${}_1A_2' = \omega$, ${}_1B_{11}' = -r$, ${}_1B_{21}' = r$ and $E_1' = 1$. The transition matrix is given by

$$\Phi'(s) = \left[sI - {}_1A_1'\right]^{-1} = \frac{1}{s+\omega}$$

The first term of the generalised Volterra series solution of the state equation is given by

$${}_1X_0 = {}_1\Phi_1'(0)\left[{}_1A_2'\ U_0 + {}_1B_{11}'\ {}_1X_0\ U_0 + {}_1B_{21}'\ U_0\ U_0\right]$$

$$= \frac{1}{\omega}\ \{\omega U_0 - r\ {}_1X_0\ U_0 + r\ U_0^2\}$$

Solving for ${}_1X_0$, gives, $\quad {}_1X_0 = U_0 \qquad (6.2.21)$

The second term is given by

$${}_1X_1(s) = {}_1\Phi_1'(s)\left[{}_1x(0) + {}_1A_2'\ U_1(s) + {}_1B_{11}'\ {}_1X_1(s)U_0 + {}_1B_{21}'\ U_0\ U_1(s)\right]$$

$$= \frac{1}{(s+\omega)}\{{}_1x(0) + \omega U_1(s) + 2r\ U_0\ U_1(s) - r\ U_0\ {}_1X_1(s) - rU_1(s)\ {}_1X_0\}$$

Solving for ${}_1X_1(s)$, gives $\displaystyle {}_1X_1(s) = \{\frac{{}_1x(0)}{(s+\omega+rU_0)} + \frac{(\omega+rU_0)}{(s+\omega+rU_0)}\ U_1(s)\}$

$$(6.2.22)$$

The third term is given by

$${}_1X_2(s_1,s_2) = {}_1\Phi_1'(s_1+s_2)\left[{}_1B_{11}'\ U_0\ {}_1X_2(s_1,s_2) + {}_1B_{11}'\ U_1(s_1)\ {}_1X_1(s_2) + {}_1B_{21}'\ U_1(s_1)U_1(s_2)\right], \text{ since } U_2(s_1,s_2)=0 \text{ and } {}_pC_{ijk}'=0.$$

$$= \frac{1}{(s_1+s_2+\omega)}\{-rU_0\ {}_1X_2(s_1,s_2) - rU_1(s_1)\ {}_1X_1(s_2) + rU_1(s_1)U_1(s_2)\}$$

Substituting for ${}_1X_0$ and ${}_1X_1(s)$ from eqns.(6.2.21) and (6.2.22) and solving, gives

$${}_1X_2(s_1,s_2) = \{\frac{rs_2\ U_1(s_1)U_1(s_2)}{(s_2+\omega+rU_0)(s_1+s_2+\omega+rU_0)} - \frac{r\ {}_1x(0)\ U_1(s_1)}{(s_2+\omega+rU_0)(s_1+s_2+\omega+rU_0)}\}$$

$$(6.2.23)$$

Similarly, the fourth term may be obtained as

$${}_1X_3(s_1,s_2,s_3) = {}_1\Phi_1'(s_1+s_2+s_3)\left[{}_1B_{11}'\ U_0\ {}_1X_3(s_1,s_2,s_3) + {}_1B_{11}'\ U_1(s_1)\ {}_1X_2(s_2,s_3)\right]$$

$$= \left\{ \frac{-r^2 \; s_3 \; U_1(s_1)U_1(s_2)U_1(s_3)}{(s_3+\omega+rU_0)(s_2+s_3+\omega+rU_0)(s_1+s_2+s_3+\omega+rU_0)} + \frac{r^2 \; {}_1x(0) \; U_1(s_1)U_1(s_2)}{(s_3+\omega+rU_0)(s_2+s_3+\omega+rU_0)(s_1+s_2+s_3+\omega+rU_0)} \right\} \qquad (6.2.24)$$

The first four terms of the generalised Volterra series expansion of the transform of the output may be obtained by noting that

$$Y_0 = {}_1X_0 = y_0 \qquad Y_1(s) = {}_1X_1(s)$$
$$Y_2(s_1,s_2) = {}_1X_2(s_1,s_2) \qquad Y_3(s_1,s_2,s_3) = {}_1X_3(s_1,s_2,s_3) \qquad (6.2.25)$$

and $y(0) = {}_1x(0)$.

If, however, the zero-order component in the input signal is assumed to be zero, then $y_0 = Y_0 = 0$ and the terms of the Volterra series expansion of the output become, from eqns.(6.2.22) to (6.2.25),

$$Y_1(s) = \left\{ \frac{y(0)}{(s+\omega)} + \frac{\omega}{(s+\omega)} \; U_1(s) \right\}$$

$$Y_2(s_1,s_2) = \left\{ \frac{rs_2 \; U_1(s_1)U_1(s_2)}{(s_2+\omega)(s_1+s_2+\omega)} - \frac{r \; y(0) \; U_1(s_1)}{(s_2+\omega)(s_1+s_2+\omega)} \right\}$$

$$Y_3(s_1,s_2,s_3) = \left\{ \frac{-r^2 s_3 \; U_1(s_1)U_1(s_2)U_1(s_3)}{(s_3+\omega)(s_2+s_3+\omega)(s_1+s_2+s_3+\omega)} + \frac{r^2 y(0) \; U_1(s_1)U_1(s_2)}{(s_3+\omega)(s_2+s_3+\omega)(s_1+s_2+s_3+\omega)} \right\} \qquad (6.2.26)$$

If the initial condition $y(0)$ is assumed to be zero, then eqns.(6.2.26) agree with the result obtained, for this system in Chapter 5, using block diagram algebra.

6.2.5 Example - Open-Loop Nonlinear System

Next, consider the system shown in Fig.6.2(a), which degenerates to an open-loop linear-square-linear system, when $\lambda = 0$. The analogue model of the system is shown in Fig.6.2(b). Let ${}_1x(t)$ and ${}_2x(t)$ be the state variables characterising the system dynamics and $\delta = \alpha\lambda$. Then, the dynamic equations, in matrix form, may be written as

$$\begin{bmatrix} {}_1\dot{x} \\ {}_2\dot{x} \end{bmatrix} = \underbrace{\begin{bmatrix} -a & -\delta \\ 0 & -b \end{bmatrix}}_{A'} \begin{bmatrix} {}_1x \\ {}_2x \end{bmatrix} + \underbrace{\begin{bmatrix} \alpha \\ 0 \end{bmatrix}}_{D'} [u] + \underbrace{\begin{bmatrix} 0 \\ \beta \end{bmatrix}}_{B'} \left[{}_1x \cdot {}_1x \right] \qquad (6.2.27a)$$

and
$$y = \begin{bmatrix} \underset{\underset{E'}{\uparrow}}{0} & 1 \end{bmatrix} \begin{bmatrix} {}_1x \\ {}_2x \end{bmatrix} \tag{6.2.27b}$$

The transition matrix is given by

$$\Phi'(s) = \left[sI - A'\right]^{-1} = \begin{bmatrix} \frac{1}{(s+a)} & \frac{-\delta}{(s+a)(s+b)} \\ 0 & \frac{1}{(s+b)} \end{bmatrix} \tag{6.2.28}$$

The generalised Volterra series solution of the state and output equations, is then obtained. The first term of the solution of the state equation is given by

$${}_pX_0 = {}_p\Phi'_i(0)\left[{}_iD'U_0 + {}_iB'_{jk}\ {}_jV'_0\ {}_kV'_0\right]$$

i.e.,
$$\begin{bmatrix} {}_1X_0 \\ {}_2X_0 \end{bmatrix} = \begin{bmatrix} \frac{1}{a} & -\frac{\delta}{ab} \\ 0 & \frac{1}{b} \end{bmatrix} \left\{ \begin{bmatrix} \alpha U_0 \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ \beta_1X_0\ {}_1X_0 \end{bmatrix} \right\}$$

Solving for ${}_1X_0$ gives
$${}_1X_0 = \left\{\frac{-ab \pm \sqrt{\left[a^2b^2 + 4\alpha\beta\delta b\ U_0\right]}}{2\delta\beta}\right\} \tag{6.2.29a}$$

and
$${}_2X_0 = \frac{\beta_1X_0\ {}_1X_0}{b} \tag{6.2.29b}$$

The second term is obtained as

$${}_pX_1(s) = {}_p\Phi'_i(s)\left[{}_ix(0) + {}_iD'U_1(s) + {}_iB'_{jk}\{{}_jV'_0\ {}_kV'_1(s) + {}_jV'_1(s)\ {}_kV'_0\}\right]$$

$$\begin{bmatrix} {}_1X_1(s) \\ {}_2X_1(s) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s+a)} & \frac{-\delta}{(s+a)(s+b)} \\ 0 & \frac{1}{(s+b)} \end{bmatrix} \left\{ \begin{bmatrix} {}_1x(0) \\ {}_2x(0) \end{bmatrix} + \begin{bmatrix} \alpha U_1(s) \\ 0 \end{bmatrix} + \begin{bmatrix} 0 \\ 2\beta_1X_0\ {}_1X_1(s) \end{bmatrix} \right\}$$

Solving for ${}_1X_1(s)$ gives

$${}_1X_1(s) = \left[\frac{(s+b)\{{}_1x(0) + \alpha U_1(s)\}}{\{(s+a)(s+b) + 2\delta\beta_1X_0\}} - \frac{{}_2x(0)\ \delta}{\{(s+a)(s+b) + 2\delta\beta_1X_0\}}\right] \tag{6.2.30a}$$

and
$${}_2X_1(s) = \left\{\frac{{}_2x(0)}{(s+b)} + \frac{2\beta_1X_0\ {}_1X_1(s)}{(s+b)}\right\} \tag{6.2.30b}$$

where ${}_1X_0$ and ${}_1X_1(s)$ are given by eqns.(6.2.29a) and (6.2.30a), respectively.

The third term is given by

$${}_pX_2(s_1,s_2) = {}_p\Phi'_i(s_1+s_2)\ {}_iB'_{jk}\left[{}_jV'_0\ {}_kV'_2(s_1,s_2) + {}_jV'_1(s_1)\ {}_kV'_1(s_2) + {}_jV'_2(s_1,s_2)\,{}_kV'_0\right]$$

i.e.,

$$\begin{bmatrix} {}_1X_2(s_1,s_2) \\ {}_2X_2(s_1,s_2) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s_1+s_2+a)} & \frac{-\delta}{(s_1+s_2+a)(s_1+s_2+b)} \\ 0 & \frac{1}{(s_1+s_2+b)} \end{bmatrix} \begin{bmatrix} 0 \\ \beta \end{bmatrix} \left[2\,{}_1X_0\;{}_1X_2(s_1,s_2) + {}_1X_1(s_1)\,{}_1X_1(s_2) \right]$$

Solving for ${}_1X_2(s_1,s_2)$ gives ${}_1X_2(s_1,s_2) = \dfrac{-\delta\beta\;{}_1X_1(s_1)\;{}_1X_1(s_2)}{\{(s_1+s_2+a)(s_1+s_2+b) + 2\delta\beta\,{}_1X_0\}}$

$$\tag{6.2.31a}$$

and $${}_2X_2(s_1,s_2) = \frac{\beta\{2\,{}_1X_0\;{}_1X_2(s_1,s_2) + {}_1X_1(s_1)\,{}_1X_1(s_2)\}}{(s_1+s_2+b)} \tag{6.2.31b}$$

where ${}_1X_0$, ${}_1X_1(s)$ and ${}_1X_2(s_1,s_2)$ are given by eqns.(6.2.29a) to (6.2.31a), respectively.

The fourth term is given by

$${}_pX_3(s_1,s_2,s_3) = {}_p\Phi'_i(s_1+s_2+s_3)\,{}_iB'_{jk}\Big[{}_jV'_0\;{}_kV'_3(s_1,s_2,s_3) + {}_jV'_1(s_1)\,{}_kV'_2(s_2,s_3) + {}_jV'_2(s_1,s_2)\,{}_kV'_1(s_3) + {}_jV'_3(s_1,s_2,s_3)\,{}_kV'_0\Big]$$

i.e.,

$$\begin{bmatrix} {}_1X_3(s_1,s_2,s_3) \\ {}_2X_3(s_1,s_2,s_3) \end{bmatrix} = \begin{bmatrix} \frac{1}{(s_1+s_2+s_3+a)} & \frac{-\delta}{(s_1+s_2+s_3+a)(s_1+s_2+s_3+b)} \\ 0 & \frac{1}{(s_1+s_2+s_3+b)} \end{bmatrix} \begin{bmatrix} 0 \\ \beta \end{bmatrix} \left[2\,{}_1X_0\;{}_1X_3(s_1,s_2,s_3) + 2\,{}_1X_1(s_1)\,{}_1X_2(s_2,s_3) \right]$$

Solving for ${}_1X_3(s_1,s_2,s_3)$ gives

$${}_1X_3(s_1,s_2,s_3) = \frac{-2\beta\delta\;{}_1X_1(s_1)\,{}_1X_2(s_2,s_3)}{\{(s_1+s_2+s_3+a)(s_1+s_2+s_3+b) + 2\beta\delta\,{}_1X_0\}} \tag{6.2.32a}$$

and $${}_2X_3(s_1,s_2,s_3) = \frac{2\delta\{{}_1X_0\;{}_1X_3(s_1,s_2,s_3) + {}_1X_1(s_1)\,{}_1X_2(s_2,s_3)\}}{(s_1+s_2+s_3+b)} \tag{6.2.32b}$$

where ${}_1X_0$, ${}_1X_1(s)$, ${}_1X_2(s_1,s_2)$ and ${}_1X_3(s_1,s_2,s_3)$ are given by eqns. (6.2.29a) to (6.2.32a), respectively.

The terms of the Volterra series solution of the output equation may then be obtained as

$$Y_0 = {}_2X_0 \qquad Y_1(s) = {}_2X_1(s)$$
$$Y_2(s_1,s_2) = {}_2X_2(s_1,s_2) \quad \text{and} \quad Y_3(s_1,s_2,s_3) = {}_2X_3(s_1,s_2,s_3) \tag{6.2.33}$$

where ${}_2X_0$ to ${}_2X_3(s_1,s_2,s_3)$ are given by eqns.(6.2.29b) to (6.2.32b), respectively and ${}_2x(0) = y(0)$.

If, however, $\lambda=0$, Fig.6.2(a) becomes an open-loop system, for which the Volterra series expansion of the output may be obtained by putting $\delta=0$ in eqns.(6.2.29b) to (6.2.32b). The results are

$$Y_0 = \frac{\alpha^2\beta U_0^2}{a^2 b}, \quad Y_1(s) = \left[\frac{y(0)}{(s+b)} + \frac{2\,\alpha\beta U_0\{{}_1x(0)+\alpha U_1(s)\}}{a(s+a)(s+b)}\right],$$

$$Y_2(s_1,s_2) = \left[\frac{\alpha^2\beta U_1(s_1)U_1(s_2)+\alpha\beta\,{}_1x(0)\{U_1(s_1)+U_1(s_2)\}+\beta\,{}_1x(0)\,{}_1x(0)}{(s_1+a)(s_2+a)(s_1+s_2+b)}\right],$$

$Y_3(s_1,s_2,s_3) = 0$, and higher order terms will be zero. (6.2.34)

Further, it is interesting to note that when the zero-order component U_0 and the initial condition $y(0)$ are equal to zero, the only non-vanishing term will be $Y_2(s_1,s_2)$, which is given by

$$Y_2(s_1,s_2) = \frac{\alpha^2\beta\; U_1(s_1)U_1(s_2)}{(s_1+a)(s_2+a)(s_1+s_2+b)}, \tag{6.2.35}$$

which is the two dimensional Laplace transform of the output of a linear -square-linear system. Thus, these two examples clearly demonstrate the validity of the state space solution developed here. The examples also show, in particular, that for nonlinear systems with multiplicative nonlinearities, the presence of a zero-order component in the input results in additional terms with a gain constant in the output of an open-loop system,whereas, in the output of a feedback system, it introduces an additional lag, both proportional to the value of the zero-order component.

6.3 Multivariable Nonlinear Systems

In practice, most of the physical systems are essentially nonlinear and may involve multiplicative interaction of various inputs and states of the system. In this section, a method is described for state space solution of such nonlinear systems. First, a more generalised form of dynamic equations are developed for characterising multi-input, multi-output nonlinear systems. Then, the solution of the dynamic equations is given. To illustrate the method, the solution is applied to analyse the response of a two-input, single-output diode-ring multiplier circuit. The results obtained here for this circuit are compared with those obtained earlier by Bansal[17], in order to validate the theoretical analysis presented here.

6.3.1 State Space Characterisation of Multivariable Systems

The dynamic equations of an R-input, Q-output nonlinear system with P state variables may be written, using Einstein summation convention and matrix partitioning, in the form as

$$
{}_p\dot{x} = \left[{}_pA^a_i \,\middle|\, {}_pA^b_L \right] \begin{bmatrix} {}_ix \\ \hline {}_Lu \end{bmatrix} + \left[{}_pB^a_{ij} \,\middle|\, {}_pB^b_{iL} \,\middle|\, {}_pB^c_{LM} \right] \begin{bmatrix} {}_ix\,{}_jx \\ \hline {}_ix\,{}_Lu \\ \hline {}_Lu\,{}_Mu \end{bmatrix}
$$

$$
+ \left[{}_pC^a_{ijk} \,\middle|\, {}_pC^b_{ijL} \,\middle|\, {}_pC^c_{iLM} \,\middle|\, {}_pC^d_{LMN} \right] \begin{bmatrix} {}_ix\,{}_jx\,{}_kx \\ \hline {}_ix\,{}_jx\,{}_Lu \\ \hline {}_ix\,{}_Lu\,{}_Mu \\ \hline {}_Lu\,{}_Mu\,{}_Nu \end{bmatrix} + \ldots\ldots \qquad (6.3.1)
$$

and

$$
{}_qy = \left[{}_qE^a_i \,\middle|\, {}_qE^b_L \right] \begin{bmatrix} {}_ix \\ \hline {}_Lu \end{bmatrix} + \left[{}_qF^a_{ij} \,\middle|\, {}_qF^b_{iL} \,\middle|\, {}_qF^c_{LM} \right] \begin{bmatrix} {}_ix\,{}_jx \\ \hline {}_ix\,{}_Lu \\ \hline {}_Lu\,{}_Mu \end{bmatrix}
$$

$$
+ \left[{}_qG^a_{ijk} \,\middle|\, {}_qG^b_{ijL} \,\middle|\, {}_qG^c_{iLM} \,\middle|\, {}_qG^d_{LMN} \right] \begin{bmatrix} {}_ix\,{}_jx\,{}_kx \\ \hline {}_ix\,{}_jx\,{}_Lu \\ \hline {}_ix\,{}_Lu\,{}_Mu \\ \hline {}_Lu\,{}_Mu\,{}_Nu \end{bmatrix} + \ldots\ldots \qquad (6.3.2)
$$

where ${}_ix(t)$ is a P x 1 state vector, ${}_Lu(t)$ is a R x 1 input vector, ${}_qy(t)$ is a Q x1 output vector, A,B,C,E,F,G are multidimensional arrays partitioned between state and input vectors, and A^a, A^b etc., are submatrices of A etc. The analogue model of the system characterised by the above equations, is shown in Fig.6.3. The solutions of eqns.(6.3.1) and (6.3.2) in terms of the initial condition vector ${}_ix(0)$ and the driving function

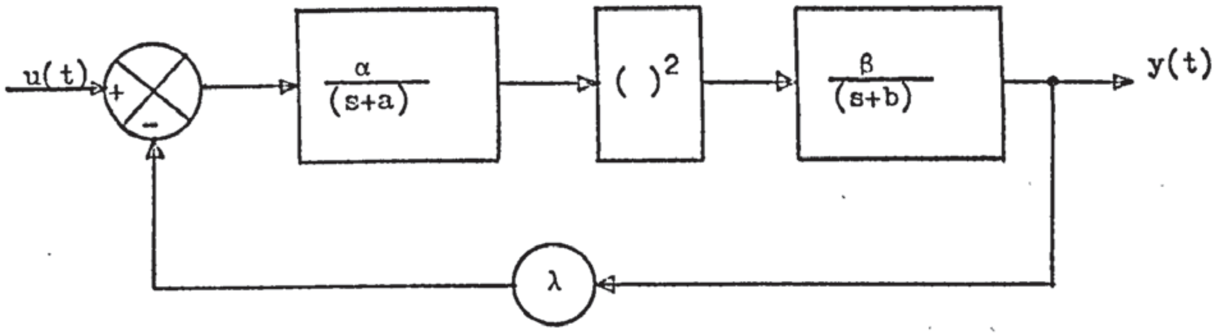


Fig.6.2(a) Open-loop nonlinear system, when λ = 0

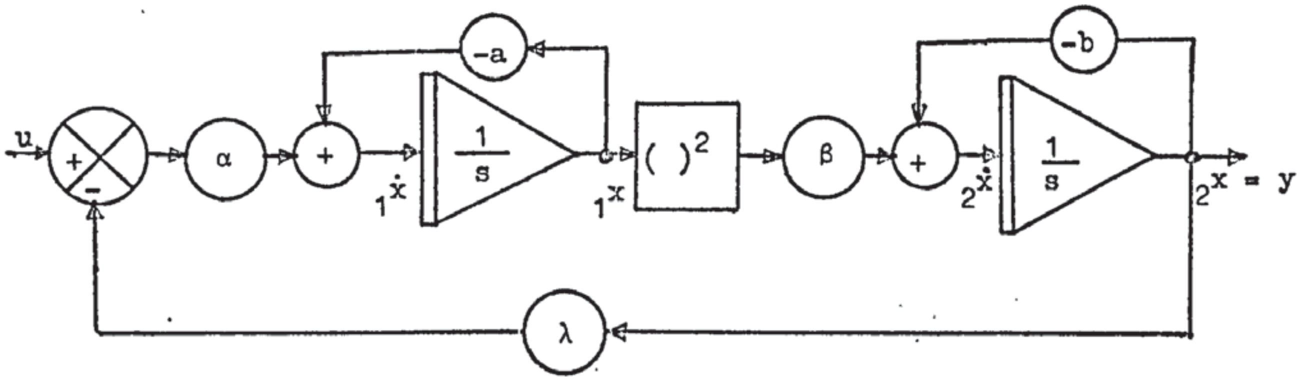


Fig.6.2(b) Analogue model of system shown in Fig.6.2(a).

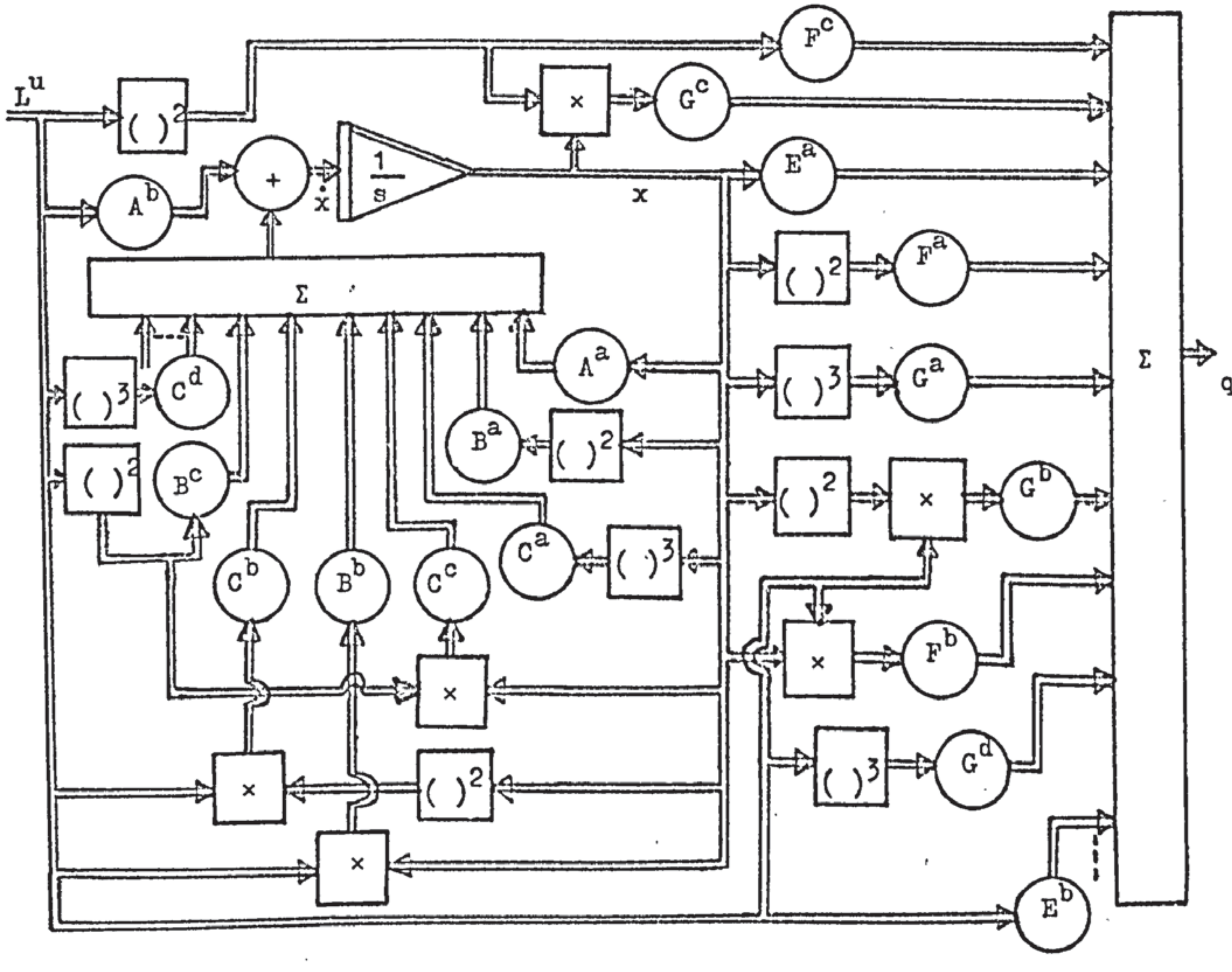


Fig.6.3 The analogue model characterising a multivariable system in state space.

vector $_Lu(t)$, are then obtained. It is assumed here, for simplicity, that the zero-order component in the input is equal to zero.

6.3.2 Solution of Dynamic Equations

To obtain the solution of the state equation, it is necessary to Laplace transform eqn.(6.3.1), which gives

$$\left[sI- A^a\right]_pX(s) = {_p}x(0)+{_p}A^b_L\,{_L}U_1(s)+\left[\begin{array}{c|c|c} {_p}B^a_{ij} & {_p}B^b_{iL} & {_p}B^c_{LM}\end{array}\right]\left[\begin{array}{c} {_i}X(s)\underset{j}{*}X(s) \\ \hline {_i}X(s)\underset{L}{*}U_1(s) \\ \hline {_L}U_1(s)\underset{M}{*}U_1(s)\end{array}\right]$$

$$+\left[\begin{array}{c|c|c|c} {_p}C^a_{ijk} & {_p}C^b_{ijL} & {_p}C^c_{iLM} & {_p}C^d_{LMN}\end{array}\right]\left[\begin{array}{c} {_i}X(s)\underset{j}{*}X(s)\underset{k}{*}X(s) \\ \hline {_i}X(s)\underset{j}{*}X(s)\underset{L}{*}U_1(s) \\ \hline {_i}X(s)\underset{L}{*}U_1(s)\underset{M}{*}U_1(s) \\ \hline {_L}U_1(s)\underset{M}{*}U_1(s)\underset{N}{*}U_1(s)\end{array}\right]$$

$$+\ldots\ldots\ldots \tag{6.3.3}$$

Following the same procedure as in section 6.2.3 and letting $\Phi^a(s) = \left[sI-A^a\right]^{-1}$, the terms of the Volterra series expansion of $_pX(s)$ may be obtained as

$${_p}X_1(s) = {_p}\Phi^a_i(s)\,{_i}x(0)+{_p}\Phi^a_i(s)\,{_i}A^b_L\,{_L}U_1(s) \tag{6.3.4}$$

$${_p}X_2(s_1,s_2) = {_p}\Phi^a_i(s_1+s_2)\left[\begin{array}{c|c|c} {_I}B^a_{ij} & {_I}B^b_{iL} & {_I}B^c_{LM}\end{array}\right]\left[\begin{array}{c} {_i}X_1(s_1)\,{_j}X_1(s_2) \\ \hline {_i}X_1(s_1)\,{_L}U_1(s_2) \\ \hline {_L}U_1(s_1)\,{_M}U_1(s_2)\end{array}\right] , \tag{6.3.5}$$

$${_p}X_3(s_1,s_2,s_3) = {_p}\Phi^a_i(s_1+s_2+s_3)$$

$$\left\{\begin{array}{l} \left[\begin{array}{c|c} {_I}B^a_{ij} & {_I}B^b_{iL}\end{array}\right]\left[\begin{array}{c} {_i}X_1(s_1)\,{_j}X_2(s_2,s_3)+{_i}X_2(s_1,s_2)\,{_j}X_1(s_3) \\ \hline {_i}X_2(s_1,s_2)\,{_L}U_1(s_3)\end{array}\right] \\ +\left[\begin{array}{c|c|c|c} {_I}C^a_{ijk} & {_I}C^b_{ijL} & {_I}C^c_{iLM} & {_I}C^d_{LMN}\end{array}\right]\left[\begin{array}{c} {_i}X_1(s_1)\,{_j}X_1(s_2)\,{_k}X_1(s_3) \\ \hline {_i}X_1(s_1)\,{_j}X_1(s_2)\,{_L}U_1(s_3) \\ \hline {_i}X_1(s_1)\,{_L}U_1(s_2)\,{_M}U_1(s_3) \\ \hline {_L}U_1(s_1)\,{_M}U_1(s_2)\,{_N}U_1(s_3)\end{array}\right]\end{array}\right\}$$

etc. $$\tag{6.3.6}$$

where $\phi^a(t)= L^{-1}\{\Phi^a(s)\}$ is the transition matrix. Assuming that the

Volterra series solution of ${}_{p}x(t)$ converges, eqns.(6.3.4) to (6.3.6) provide a complete solution to the state eqn.(6.3.1). It is easily observed that the higher order terms of the Volterra series may be obtained from the lower order terms and the state transition matrix. It may also be noted that, in eqn.(6.3.4), which represents the solution of the linear part of the state equation, the effects of the initial condition vector and the input vector, on the state of the system, are independent whereas eqns.(6.3.5) and (6.3.6) show, explicitly, the interaction between the state vector, the initial condition vector and the input vector.

Substituting the Volterra series expansions of ${}_{q}Y(s)$ and ${}_{i}X(s)$ into the transform of the output equation (6.3.2) and equating terms of equal order results in

$$ {}_{q}Y_1(s) = \left[{}_{q}E^{a}_{i}\;{}_{i}X_1(s) + {}_{q}E^{b}_{L}\;{}_{L}U_1(s)\right] \tag{6.3.7}$$

$$ {}_{q}Y_2(s_1,s_2) = \Big[{}_{q}E^{a}_{i}\;{}_{i}X_2(s_1,s_2) + {}_{q}F^{a}_{ij}\;{}_{i}X_1(s_1)\,{}_{j}X_1(s_2) + {}_{q}F^{b}_{iL}\;{}_{i}X_1(s_1)\,{}_{L}U_1(s_2) + {}_{q}F^{c}_{LM}\;{}_{L}U_1(s_1)\,{}_{M}U_1(s_2)\Big] \tag{6.3.8}$$

$$ \begin{aligned} {}_{q}Y_3(s_1,s_2,s_3) &= {}_{q}E^{a}_{i}\;{}_{i}X_3(s_1,s_2,s_3) + \left[\,{}_{q}F^{a}_{ij} \;\vdots\; {}_{q}F^{b}_{iL}\right] \left[\begin{array}{c} {}_{i}X_1(s_1)\,{}_{j}X_2(s_2,s_3) + {}_{i}X_2(s_1,s_2)\,{}_{j}X_1(s_3) \\ \hdashline {}_{i}X_2(s_1,s_2)\,{}_{L}U_1(s_3) \end{array}\right] \\ &\quad + \left[\,{}_{q}G^{a}_{ijk} \;\vdots\; {}_{q}G^{b}_{ijL} \;\vdots\; {}_{q}G^{c}_{iLM} \;\vdots\; {}_{q}G^{d}_{LMN}\right] \left[\begin{array}{c} {}_{i}X_1(s_1)\,{}_{j}X_1(s_2)\,{}_{k}X_1(s_3) \\ \hdashline {}_{i}X_1(s_1)\,{}_{j}X_1(s_2)\,{}_{L}U_1(s_3) \\ \hdashline {}_{i}X_1(s_1)\,{}_{L}U_1(s_2)\,{}_{M}U_1(s_3) \\ \hdashline {}_{L}U_1(s_1)\,{}_{M}U_1(s_2)\,{}_{N}U_1(s_3) \end{array}\right] \end{aligned}$$

etc. (6.3.9)

where ${}_{i}X_1(s_1)$, ${}_{i}X_2(s_1,s_2)$, ${}_{i}X_3(s_1,s_2,s_3)$, are given by eqns.(6.3.4) to (6.3.6). Eqn.(6.3.7) is a well known result in linear system theory[83], and gives the output in terms of the state transition matrix. Then, eqns. (6.3.8) and (6.3.9) represent the outputs of the second and third-order kernels, respectively. Assuming that the Volterra series expansion of the output is convergent, eqns.(6.3.7) to (6.3.9) represent the state-space solution to output equation (6.3.2).

6.3.3 Example - Diode Ring Multiplier

The basic configuration of a diode-ring multiplier circuit[17] is shown in Fig.6.4, in which 2_1u and 2_2u are the two input voltage sources. It is assumed here, for simlicity, that the current-voltage characteristics of the four diodes in the ring are identical. The current i through the inductive load is given by

$$i = i_1 - i_2 + i_3 - i_4 \tag{6.3.10}$$

where the current i_k,(k=1,2,3,4), through a diode d_k is related to its impressed voltage ${}_ku_d$ by the equation

$$i_k = A(e^{\alpha_k u_d} - 1) \quad , k = 1,2,3,4 \tag{6.3.11}$$

$$\begin{aligned} &\text{where } {}_1u_d = -{}_1u - {}_2u - y \qquad {}_2u_d = {}_1u - {}_2u + y \\ &{}_3u_d = {}_1u + {}_2u - y \quad \text{and} \quad {}_4u_d = -{}_1u + {}_2u + y \end{aligned} \tag{6.3.12}$$

where y is the output voltage across the load circuit, given by

$$y = J_1 \otimes i \quad , \tag{6.3.13}$$

where J_1 = (L d/dt + R), so that $J_1(s)$ = (sL + R) and R and L are the resistance and inductance of the load circuit, respectively.

Substituting eqns.(6.3.11) and (6.3.12) into eqn.(6.3.10) yields the current i through the load, as

$$i = A\{e^{\alpha(-{}_1u - {}_2u - y)} - e^{\alpha({}_1u - {}_2u + y)} + e^{\alpha({}_1u + {}_2u - y)} - e^{\alpha(-{}_1u + {}_2u + y)}\} \tag{6.3.14}$$

which then gives an implicit relationship between y and ${}_1u$ and ${}_2u$. To obtain an explicit expression for y in terms of ${}_1u$ and ${}_2u$, it is necessary to approximate e^θ by first few terms of its series expansion, as

$$e^\theta = (1 + \theta + \frac{\theta^2}{2} + \frac{\theta^3}{6} + \frac{\theta^4}{24}) \tag{6.3.15}$$

The above equation is valid because, it is assumed here that the multiplier circuit is characterised by the Volterra kernels upto fourth order only, and the higher order terms in the series expansion of e^θ do not contribute to Volterra kernels upto fourth order. Let ${}_1x = y$, be the state variable characterising the multiplier circuit. Substituting this and eqn.(6.3.15) into (6.3.14), yields

$$i = A\{-4\alpha\,{}_1x + 4\alpha^2\,{}_1u\,{}_2u - 2\alpha^3\,{}_1u^2\,{}_1x - 2\alpha^3\,{}_2u^2\,{}_1x - \frac{2\alpha^3}{3}\,{}_1x^3 + \frac{2\alpha^4}{3}\,{}_1u^3\,{}_2u$$
$$+ \frac{2\alpha^4}{3}\,{}_1u\,{}_2u^3 + 2\alpha^4\,{}_1u\,{}_2u\,{}_1x^2\} \qquad (6.3.16)$$

Then, substituting eqn.(6.3.16) into eqn.(6.3.13) and rearranging, the dynamic equations characterising the multiplier circuit may be written in the matrix form as

$$\left[{}_1\dot{x}\right] = \begin{array}{c}\left[\begin{array}{c|cc} -\beta & 0 & 0 \end{array}\right] \\ \begin{array}{cc} A^a & \quad A^b \end{array}\end{array} \left[\begin{array}{c} {}_1x \\ \hline {}_1u \\ {}_2u \end{array}\right] + \begin{array}{c}\left[\begin{array}{cc|cc|c} 0 & 0 & 0 & 0 & \frac{\alpha}{L}J_1 \end{array}\right] \\ \begin{array}{ccc} B^a & \quad B^b & \quad B^c \end{array}\end{array} \left[\begin{array}{c} {}_1x\,{}_1x \\ {}_1x\,{}_2x \\ \hline {}_1x\,{}_1u \\ {}_1x\,{}_2u \\ \hline {}_1u\,{}_2u \end{array}\right]$$

$$+ \begin{array}{c}\left[\begin{array}{c|cc|cc|cc} -\frac{\alpha^2}{6L}J_1 & 0 & 0 & -\frac{\alpha^2}{2L}J_1 & -\frac{\alpha^2}{2L}J_1 & 0 & 0 \end{array}\right] \\ \begin{array}{cccc} C^a & \qquad C^b & \qquad C^c & \qquad\qquad C^d \end{array}\end{array} \left[\begin{array}{c} {}_1x\,{}_1x\,{}_1x \\ \hline {}_1x\,{}_1x\,{}_1u \\ {}_1x\,{}_1x\,{}_2u \\ \hline {}_1x\,{}_1u\,{}_1u \\ {}_1x\,{}_2u\,{}_2u \\ \hline {}_1u\,{}_1u\,{}_1u \\ {}_2u\,{}_2u\,{}_2u \end{array}\right] \qquad (6.3.17)$$

$$+ \begin{array}{c}\left[\begin{array}{c|cc|c|cc|cc} 0 & 0 & 0 & \frac{\alpha^3}{2L}J_1 & 0 & 0 & \frac{\alpha^3}{6L}J_1 & \frac{\alpha^3}{6L}J_1 \end{array}\right] \\ \begin{array}{ccccc} D^a & \quad D^b & \quad D^c & \qquad D^d & \qquad D^e \end{array}\end{array} \left[\begin{array}{c} {}_1x\,{}_1x\,{}_1x\,{}_1x \\ \hline {}_1x\,{}_1x\,{}_1x\,{}_1u \\ {}_1x\,{}_1x\,{}_1x\,{}_2u \\ \hline {}_1x\,{}_1x\,{}_1u\,{}_2u \\ \hline {}_1x\,{}_1u\,{}_1u\,{}_1u \\ {}_1x\,{}_2u\,{}_2u\,{}_2u \\ \hline {}_1u\,{}_1u\,{}_1u\,{}_2u \\ {}_1u\,{}_2u\,{}_2u\,{}_2u \end{array}\right]$$

and

$$\left[y\right] = \begin{array}{c} \left[\begin{array}{c|cc} 1 & 0 & 0 \end{array}\right] \\ \uparrow \quad \uparrow \\ E^a \quad E^b \end{array} \begin{bmatrix} {}_1x \\ \hline {}_1u \\ {}_2u \end{bmatrix} \tag{6.3.18}$$

where $\beta = (\frac{4\ \alpha AR + 1}{4\ \alpha AL})$. The solution of the dynamic equations is then obtained using eqns.(6.3.4) to (6.3.9). First, the transition matrix is derived, as

$$\Phi^a(s) = \left[sI - A^a\right]^{-1} = \frac{1}{(s+\beta)} \tag{6.3.19}$$

Since the input signals ${}_1u$ and ${}_2u$ do not have any zero-order components, ${}_1U_0 = {}_2U_0 = 0$ and also since $B^a = B^b = 0$, the zero-order component vector ${}_pX_0 = 0$ i.e., ${}_1X_0 = 0$. The first-order term in the Volterra series solution of the state equation is given by

$${}_pX_1(s) = {}_p\Phi^a_i(s)_i x(0) + {}_p\Phi^a_i(s)_i A^b_L\ {}_LU_1(s)$$

which gives ${}_1X_1(s) = \dfrac{{}_1x(0)}{(s+\beta)}$, since $A^b=0$. (6.3.20)

The second-order term is obtained as

$${}_pX_2(s_1,s_2) = {}_p\Phi^a_i(s_1+s_2)\ {}_iB^c_{LM}\ {}_LU_1(s_1)_MU_1(s_2), \text{ since } B^a=0,\ B^b=0,$$

$$\text{which gives} \quad {}_1X_2(s_1,s_2) = \frac{\alpha J_1(s_1+s_2)_1U_1(s_1)_2U_1(s_2)}{L(s_1+s_2+\beta)} \tag{6.3.21}$$

From eqn.(6.3.6), the third-order term is obtained as

$${}_pX_3(s_1,s_2,s_3) = {}_p\Phi^a_i(s_1+s_2+s_3)\Big[{}_iC^a_{jkl}\ {}_jX_1(s_1)_kX_1(s_2)_lX_1(s_3)$$
$$+ {}_iC^c_{jLM}\ {}_jX_1(s_1)_LU_1(s_2)_MU_1(s_3)\Big] ,$$

since $B^a=0$, $B^b=0$, $C^b=0$, $C^d=0$, which gives

$$\begin{aligned} {}_1X_3(s_1,s_2,s_3) = &-\frac{\alpha^2\ J_1(s_1+s_2+s_3)_1X_1(s_1)_1X_1(s_2)_1X_1(s_3)}{6L(s_1+s_2+s_3+\beta)} \\ &-\frac{\alpha^2\ J_1(s_1+s_2+s_3)_1X_1(s_1)\{{}_1U_1(s_2)_1U_1(s_3)+{}_2U_1(s_2)_2U_1(s_3)\}}{2L(s_1+s_2+s_3+\beta)} \end{aligned} \tag{6.3.22}$$

The fourth-order term is given by

$$
\begin{aligned}
&{}_pX_4(s_1,s_2,s_3,s_4)\\
&\quad = {}_p\Phi^a_i(s_1+s_2+s_3+s_4)\{{}_iC^a_{jkl}\Big[{}_jX_1(s_1)_kX_1(s_2)_lX_2(s_3,s_4)\\
&\quad + {}_jX_1(s_1)_kX_2(s_2,s_3)_lX_1(s_4) + {}_jX_2(s_1,s_2)_kX_1(s_3)_lX_1(s_4)\Big]\\
&\quad + {}_iC^c_{jLM}\ {}_jX_2(s_1,s_2)_LU_1(s_3)_MU_1(s_4) + {}_iD^c_{jkLM}\ {}_jX_1(s_1)_kX_1(s_2)_LU_1(s_3)_MU_1(s_4)\\
&\quad + {}_iD^e_{LMNP}\ {}_LU_1(s_1)_MU_1(s_2)_NU_1(s_3)_PU_1(s_4)\}
\end{aligned}
$$

since $B^a = 0$, $B^b = 0$, $C^b = 0$, $C^d = 0$, $D^a = 0$, $D^b = 0$ and $D^d = 0$. Further, ${}_pX_4(s_1,s_2,s_3,s_4)$ is symmetrical in Laplace variables s_1 to s_4, and hence is given by

$$
\begin{aligned}
&{}_1X_4(s_1,s_2,s_3,s_4)\\
&\quad = -\frac{\alpha^2\ J_1(s_1+s_2+s_3+s_4)}{6L(s_1+s_2+s_3+s_4+\beta)}\Big[3\,{}_1X_1(s_1)_1X_1(s_2)_1X_2(s_3,s_4)\\
&\quad + 3\,{}_1X_2(s_1,s_2)\{{}_1U_1(s_3)_1U_1(s_4) + {}_2U_1(s_3)_2U_1(s_4)\}\\
&\quad - 3\alpha\ {}_1X_1(s_1)_1X_1(s_2)_1U_1(s_3)_2U_1(s_4) - \alpha\ {}_1U_1(s_1)_1U_1(s_2)_1U_1(s_3)_2U_1(s_4)\\
&\quad - \alpha\,{}_1U_1(s_1)_2U_1(s_2)_2U_1(s_3)_2U_1(s_4)\Big]
\end{aligned}
$$

Thus, the terms of the Volterra series solution of the output equation, are given by

$$
\begin{aligned}
&Y_1(s) = {}_1X_1(s) \qquad\qquad Y_2(s_1,s_2) = {}_1X_2(s_1,s_2) \qquad (6.3.24)\\
&Y_3(s_1,s_2,s_3) = {}_1X_3(s_1,s_2,s_3) \quad \text{and} \quad Y_4(s_1,s_2,s_3,s_4) = {}_1X_4(s_1,s_2,s_3,s_4)
\end{aligned}
$$

where ${}_1X_1(s)$ to ${}_1X_4(s_1,s_2,s_3,s_4)$ are given by eqns.(6.3.20) to (6.3.23), respectively.

If the initial conditions ${}_1x(0) = y(0)$ is equal to zero, then eqns. (6.3.24) become

$$
Y_1(s) = 0\ ,\quad Y_2(s_1,s_2) = \frac{\alpha\{(s_1+s_2)L + R\}\,{}_1U_1(s_1)_2U_1(s_2)}{L(s_1+s_2+\beta)},\ Y_3(s_1,s_2,s_3) = 0,
$$

$$
\begin{aligned}
Y_4(s_1,s_2,s_3,s_4) = {}& \frac{-\alpha^2\{(s_1+s_2+s_3+s_4)L + R\}}{24AL^2(s_1+s_2+s_3+s_4+\beta)} \qquad (6.3.25)\\
& \Bigg[\frac{\big[8\alpha A\{(s_3+s_4)L + R\} - 1\big]\,{}_1U_1(s_1)_1U_1(s_2)_1U_1(s_3)_2U_1(s_4)}{(s_3+s_4+\beta)}\\
& + \frac{\big[8\alpha A\{(s_1+s_2)L + R\} - 1\big]\,{}_1U_1(s_1)_2U_1(s_2)_2U_1(s_3)_2U_1(s_4)}{(s_1+s_2+\beta)}\Bigg]
\end{aligned}
$$

It may be noted that the second-order term $Y_2(s_1,s_2)$ agrees with the result obtained by Bansal[17] for diode ring multiplier, but the fourth order terms, $Y_4(s_1,s_2,s_3,s_4)$ do not agree. The reason for this discrepancy is that, in the earlier analysis, the contribution of θ^4 term in eqn.(6.3.15), to $Y_4(s_1,s_2,s_3,s_4)$, has not been taken into account.

6.4 Multidimensional Laplace Transform Kernels

It is of interest to investigate the relationship between the multidimensional transform kernels and the transition matrix. In this section, a method of deriving and synthesising the multidimensional Laplace transform kernels from the solution of output equation is described.

6.4.1 Relationship between Transition Matrix and Volterra Kernels of a Nonlinear System

Consider an R-input, Q-output continuous nonlinear system with P state variables. The terms of the Volterra series expansion of the transform of the output vector may then be written in terms of multidimensional Laplace transform kernels, as

$${}_qY_1(s) = {}_qW_{1L}(s)\,{}_LU_1(s) \tag{6.4.1}$$

$${}_qY_2(s_1,s_2) = {}_qW_{2LM}(s_1,s_2)\,{}_LU_1(s_1)\,{}_MU_1(s_2) \tag{6.4.2}$$

$${}_qY_3(s_1,s_2,s_3) = {}_qW_{3LMN}(s_1,s_2,s_3)\,{}_LU_1(s_1)\,{}_MU_1(s_2)\,{}_NU_1(s_3) \tag{6.4.3}$$

and

$${}_qY_n(s_1,s_2,\ldots,s_n) = {}_qW_{nLMN\ldots K}(s_1,\ldots,s_n)\,{}_LU_1(s_1)\,{}_MU_1(s_2)\,{}_NU_1(s_3)\ldots{}_KU_1(s_n)$$
$$n > 3 \tag{6.4.4}$$

where ${}_qW_{nLMN\ldots K}(s_1,\ldots,s_n)$ is the multidimensional Laplace transform of n^{th} order Volterra kernel which completely characterise the system output ${}_qy(t)$. In this section, an attempt is made to obtain only the first three kernels, ${}_qW_{1L}(s)$, ${}_qW_{2LM}(s_1,s_2)$, ${}_qW_{3LMN}(s_1,s_2,s_3)$, of the multivariable system and synthesise them in terms of linear state transition matrix $\Phi^a(s)$.

If the system is described by the dynamic equations (6.3.1) and

(6.3.2), then the first three terms of the Volterra series expansion of the output may be obtained from eqns.(6.3.4) to (6.3.9) as

$$ {}_qY_1(s) = {}_qE^a_I\ {}_I\Phi^a_J(s)_J x(0) + \{{}_qE^a_I\ {}_I\Phi^a_J(s)_J A^b_L + {}_qE^b_L\}_L U_1(s) \tag{6.4.5} $$

$$ \begin{aligned} {}_qY_2(s_1,s_2) = {} & \{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2)_J B^a_{ij} + {}_qF^a_{ij}\}_i X_1(s_1)_j X_1(s_2) \\ & + \{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2)_J B^b_{iL} + {}_qF^b_{iL}\}_i X_1(s_1)_L U_1(s_2) \\ & + \{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2)_J B^c_{LM} + {}_qF^c_{LM}\}_L U_1(s_1)_M U_1(s_2) \end{aligned} \tag{6.4.6} $$

$$ \begin{aligned} {}_qY_3(s_1,s_2,s_3) = {} & \Big[\{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^a_{iK} + {}_JF^a_{iK}\}_K \Phi^a_w(s_2+s_3)_w B^a_{jk} \\ & + \{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^a_{kw} + {}_qF^a_{kw}\}_w \Phi^a_K(s_1+s_2)_K B^a_{ij} \\ & + \{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J C^a_{ijk} + {}_qG^a_{ijk}\}\Big]_i X_1(s_1)_j X_1(s_2)_k X_1(s_3) \\ & + \Big[\{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^a_{iK} + {}_qF^a_{iK}\}_K \Phi^a_w(s_2+s_3)_w B^b_{jL} \\ & + \{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^b_{KL} + {}_qF^b_{KL}\}_K \Phi^a_w(s_1+s_2)_w B^a_{ij} \\ & + {}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J C^b_{ijL} + {}_qG^b_{ijL}\Big]_i X_1(s_1)_j X_1(s_2)_L U_1(s_3) \\ & + \Big[{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^a_{kw} + {}_qF^a_{kw}\Big]_w \Phi^a_K(s_1+s_2)_K B^b_{iL}\ {}_iX_1(s_1) \\ & \qquad {}_LU_1(s_2)_k X_1(s_3) \\ & + \Big[\{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^a_{iK} + {}_qF^a_{iK}\}_K \Phi^a_w(s_2+s_3)_w B^c_{LM} \\ & + \{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^b_{KM} + {}_qF^b_{KM}\}_K \Phi^a_w(s_1+s_2)_w B^b_{iL} \\ & + {}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J C^c_{iLM} + {}_qG^c_{iLM}\Big]_i X_1(s_1)_L U_1(s_2)_M U_1(s_3) \\ & + \Big[{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^a_{kw} + {}_qF^a_{kw}\Big]_w \Phi^a_K(s_1+s_2)_K B^c_{LM}\ {}_LU_1(s_1) \\ & \qquad {}_MU_1(s_2)_k X_1(s_3) \\ & + \Big[\{{}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J B^b_{KN} + {}_qF^b_{KN}\}_K \Phi^a_w(s_1+s_2)_w B^c_{LM} \\ & + {}_qE^a_I\ {}_I\Phi^a_J(s_1+s_2+s_3)_J C^d_{LMN} + {}_qG^d_{LMN}\Big]_L U_1(s_1)_M U_1(s_2)_N U_1(s_3) \end{aligned} \tag{6.4.7} $$

where ${}_iX_1(s\ )$ is given by eqn.(6.3.4).

6.4.2 Synthesis of Multidimensional Laplace Transform Kernels

The Volterra series solution of the output equation characterising the multivariable system is given by eqns.(6.4.5) to (6.4.7). Now, for obtaining the multidimensional Laplace transform kernels, it is to be assumed, as in the linear case, that the initial condition vector ${}_ix(0)$ is zero. Then, for ${}_ix(0) = 0$, substituting eqn.(6.3.4) into eqns.(6.4.5) to (6.4.7) and comparing the resulting equations with eqns.(6.4.1) to (6.4.3), respectively, the multidimensional Laplace transform kernels of the system may be obtained as

$$ {}_qW_{1L}(s) = \left[{}_qE^a_I \; {}_I\Phi^a_J(s)\, {}_JA^b_L + {}_qE^b_L \right] \tag{6.4.8} $$

$$ \begin{aligned} {}_qW_{2LM}(s_1,s_2) = \Big[& \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2)\, {}_JB^a_{ij} + {}_qF^a_{ij} \} \, {}_i\Phi^a_K(s_1)\, {}_KA^b_L \; {}_j\Phi^a_w(s_2)\, {}_wA^b_M \\ & + \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2)\, {}_JB^b_{iL} + {}_qF^b_{iL} \} \, {}_i\Phi^a_K(s_1)\, {}_KA^b_M \\ & + {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2)\, {}_JB^c_{LM} + {}_qF^c_{LM} \Big] \end{aligned} \tag{6.4.9} $$

$$ \begin{aligned} & {}_qW_{3LMN}(s_1,s_2,s_3) \\ & = \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2+s_3)\, {}_JB^a_{iK} + {}_qF^a_{iK} \} \Big[{}_i\Phi^a_S(s_1)\, {}_SA^b_L \; {}_K\Phi^a_w(s_2+s_3) \\ & \quad \{ {}_wB^a_{jk} \; {}_j\Phi^a_T(s_2)\, {}_TA^b_M \; {}_k\Phi^a_Z(s_3)\, {}_ZA^b_N + {}_wB^b_{jN} \; {}_j\Phi^a_T(s_2)\, {}_TA^b_M + {}_wB^c_{MN} \} \\ & \quad + {}_i\Phi^a_w(s_1+s_2) \{ {}_wB^a_{jk} \; {}_j\Phi^a_T(s_1)\, {}_TA^b_L \; {}_k\Phi^a_Z(s_2)\, {}_ZA^b_M + {}_wB^b_{jM} \; {}_j\Phi^a_T(s_1)\, {}_TA^b_L + {}_wB^c_{LM} \} \\ & \qquad {}_K\Phi^a_S(s_3)\, {}_SA^b_N \Big] \\ & + \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2+s_3)\, {}_JB^b_{KN} + {}_qF^b_{KN} \} \Big[{}_K\Phi^a_w(s_1+s_2) \\ & \quad \{ {}_wB^a_{ij} \; {}_i\Phi^a_S(s_1)\, {}_SA^b_L \; {}_j\Phi^a_T(s_2)\, {}_TA^b_M + {}_wB^b_{iM} \; {}_i\Phi^a_S(s_1)\, {}_SA^b_L + {}_wB^c_{LM} \} \Big] \\ & + \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2+s_3)\, {}_JC^a_{ijk} + {}_qG^a_{ijk} \} \, {}_i\Phi^a_S(s_1)\, {}_SA^b_L \; {}_j\Phi^a_T(s_2)\, {}_TA^b_M \; {}_k\Phi^a_Z(s_3)\, {}_ZA^b_N \\ & + \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2+s_3)\, {}_JC^b_{ijL} + {}_qG^b_{ijL} \} \, {}_i\Phi^a_S(s_1)\, {}_SA^b_L \; {}_j\Phi^a_T(s_2)\, {}_TA^b_M \\ & + \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2+s_3)\, {}_JC^c_{iLM} + {}_qG^c_{iLM} \} \, {}_i\Phi^a_S(s_1)\, {}_SA^b_L \\ & + \{ {}_qE^a_I \; {}_I\Phi^a_J(s_1+s_2+s_3)\, {}_JC^d_{LMN} + {}_qG^d_{LMN} \} \end{aligned} \tag{6.4.10} $$

where $\Phi^a(s)$ is the state transition matrix.

The higher order transform kernels may be similarly obtained. Eqn. (6.4.8) is the transfer function matrix of the linear system[83] described by the first term of the dynamic eqns.(6.3.1) and (6.3.2). Then, following the synthesis procedure analogous to the one described in Chapter 5, ${}_qW_{1L}(s)$ and ${}_qW_{2LM}(s_1,s_2)$ are realised as shown in Figs.6.5(a) and (b), respectively. The third-order kernel ${}_qW_{3LMN}(s_1,s_2,s_3)$ may be similarly synthesised.

6.4.3 Example - Diode Ring Multiplier

The solution of the dynamic equations characterising the two-input, single-output diode ring multiplier circuit was given in section 6.3. Now, the multidimensional Volterra kernels characterising the diode ring multiplier are derived using the theory developed here.

The transfer function matrix of the linear kernel is given by eqn. (6.4.8), where q=1 and L = 1 and 2, and $\Phi^a(s) = \frac{1}{(s+\beta)}$. Thus,

$$ {}_1W_{1L}(s) = {}_1E^a_i\ {}_i\Phi^a_j(s)\,{}_jA^b_L + {}_1E^b_L \tag{6.4.11}$$

Since $E^a = 1$, $A^b=0$ and $E^b=0$, we obtain

$$ {}_1W_{11}(s) = 0 \text{ and } {}_1W_{12}(s) = 0 \tag{6.4.12}$$

The second-order transform kernel is obtained from eqn.(6.4.9) as

$$ \begin{aligned} {}_1W_{2LM}(s_1,s_2) &= {}_1E^a_i\ {}_i\Phi^a_j(s_1+s_2)\,{}_jB^c_{LM}\,, \qquad L=1,\ M=2 \\ &= \frac{\alpha J_1(s_1+s_2)}{L(s_1+s_2+\beta\ )} \end{aligned} \tag{6.4.13}$$

It may be noted that, ${}_1W_{212}$ is only present and all the other second-order kernels, such as ${}_1W_{211}$, ${}_1W_{222}$ etc., are equal to zero.

From eqn.(6.4.10), the third-order kernel is given by

$$ {}_1W_{3LMN}(s_1,s_2,s_3) = 0\ , \text{ since } A^b=B^a=B^b=C^b=C^d=E^b=F^a=F^b=F^c=G^a=G^b=G^c=G^d=0, $$

which means that all third-order kernels, such as ${}_1W_{3111}$, ${}_1W_{3222}$, ${}_1W_{3122}$, ${}_1W_{3211}$ etc., are equal to zero.

The fourth-order kernel may be similarly obtained, as

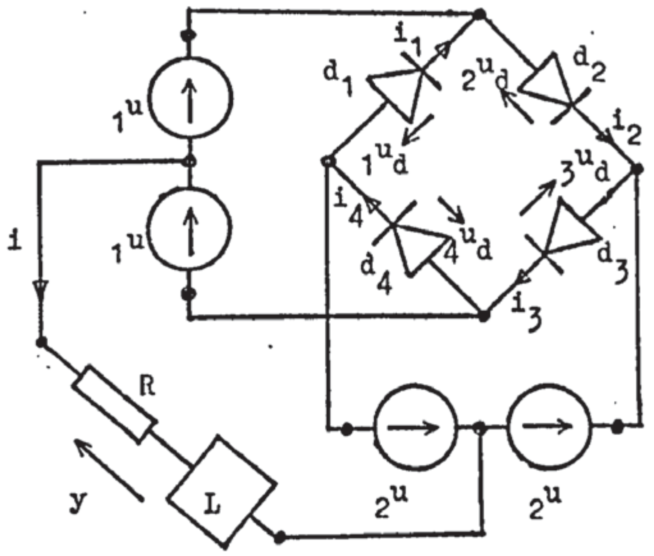


Fig.6.4 Diode-ring multiplier circuit.

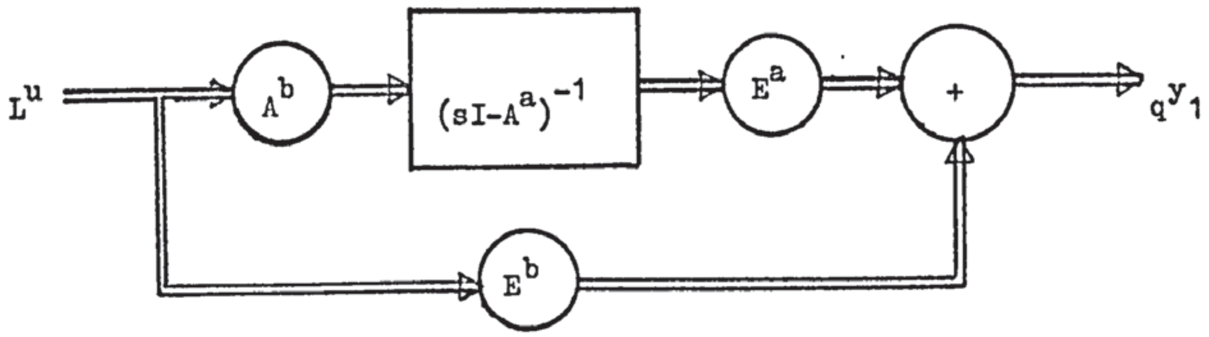


Fig.6.5(a) First-order kernel , ${}_qW_{1L}(s)$.

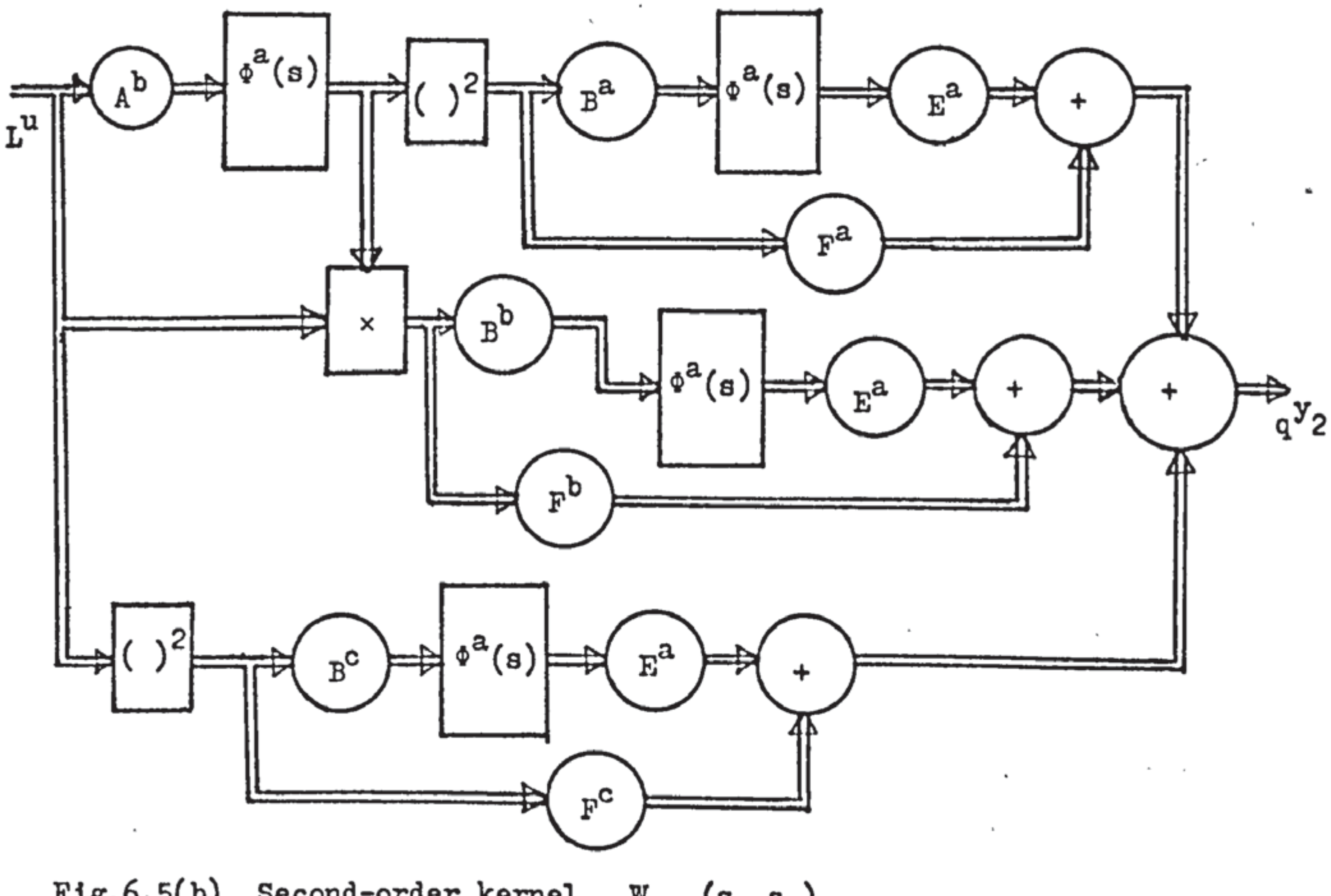


Fig.6.5(b) Second-order kernel, ${}_qW_{2LM}(s_1,s_2)$.

$$_1W_{4LMNP}(s_1,s_2,s_3,s_4) = {}_qE^a_i\ {}_i\Phi^a_j(s_1+s_2+s_3+s_4)\Big[{}_jC^c_{kNP}\ {}_k\Phi^a_l(s_1+s_2)\,{}_lB^c_{LM} + {}_jD^e_{LMNP}\Big] ,$$

which gives

$$_1W_{41112}(s_1,s_2,s_3,s_4) = \frac{\alpha^3 J_1(s_1+s_2+s_3+s_4)}{6L(s_1+s_2+s_3+s_4+\beta)} - \frac{\alpha^3 J_1(s_1+s_2+s_3+s_4)J_1(s_3+s_4)}{2L^2(s_1+s_2+s_3+s_4+\beta)(s_3+s_4+\beta)}$$

and

$$_1W_{41222}(s_1,s_2,s_3,s_4) = \frac{\alpha^3 J_1(s_1+s_2+s_3+s_4)}{6L(s_1+s_2+s_3+s_4+\beta)} - \frac{\alpha^3 J_1(s_1+s_2+s_3+s_4)J_1(s_1+s_2)}{2L^2(s_1+s_2+s_3+s_4+\beta)(s_1+s_2+\beta)}$$

(6.4.14)

Since the kernels are symmetric, it may be noted that $_1W_{41112}(s_1,s_2,s_3,s_4) = {}_1W_{41222}(s_1,s_2,s_3,s_4)$. Then, the non-vanishing kernels $_1W_{212}$, $_1W_{41112}$ and $_1W_{41222}$ may be synthesised as shown in Figs. 6.6(a) and 6.6(b), respectively.

6.5 Conclusions

The dynamic equations developed here may be used to characterise the nonlinear systems, having multiplicative, functional or polynomial type nonlinearities, completely in state space. However, the nonlinear systems with zero-order component in the input signal can be characterised by the generalised dynamic equations. The solution of the dynamic equations in all cases shows, explicitly, the influence of the initial conditions and the driving functions on the state of the system and also indicates how the transition matrix operates on each of the terms of the Volterra series expansion, to obtain the state and the output of the system in terms of the initial conditions and the input signal.

From the solution of the dynamic equations characterising the multivariable system, an explicit input-output relationship has been established. Using this relationship, the multidimensional Laplace transform kernels may be derived and synthesised in terms of the transition matrix. The validity of the method has been demonstrated for both single variable and multivariable systems by analysing the response of the direction dependent system and the diode ring multiplier circuit, respectively.

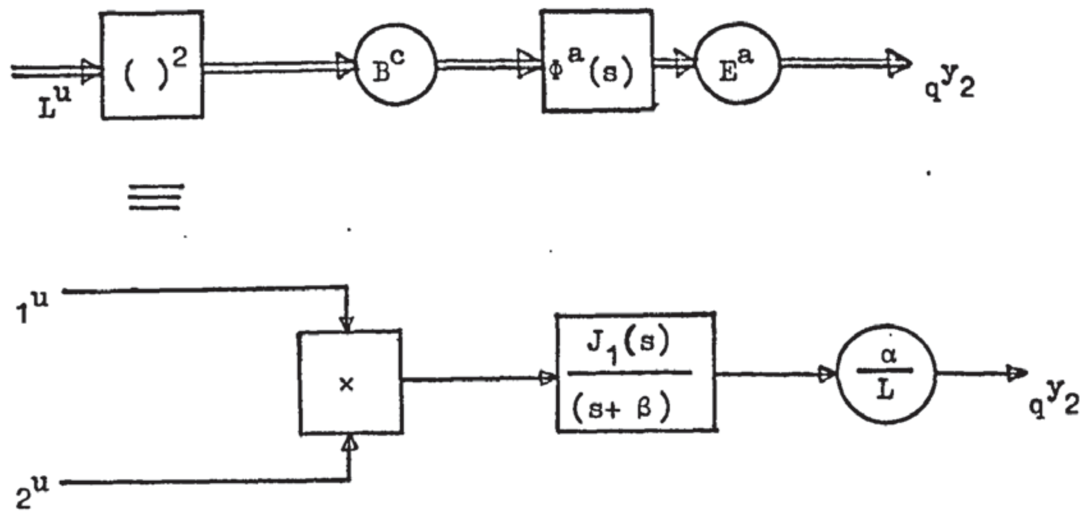


Fig.6.6(a) Second-order kernel, $_1W_{212}$ of the diode-ring multiplier.

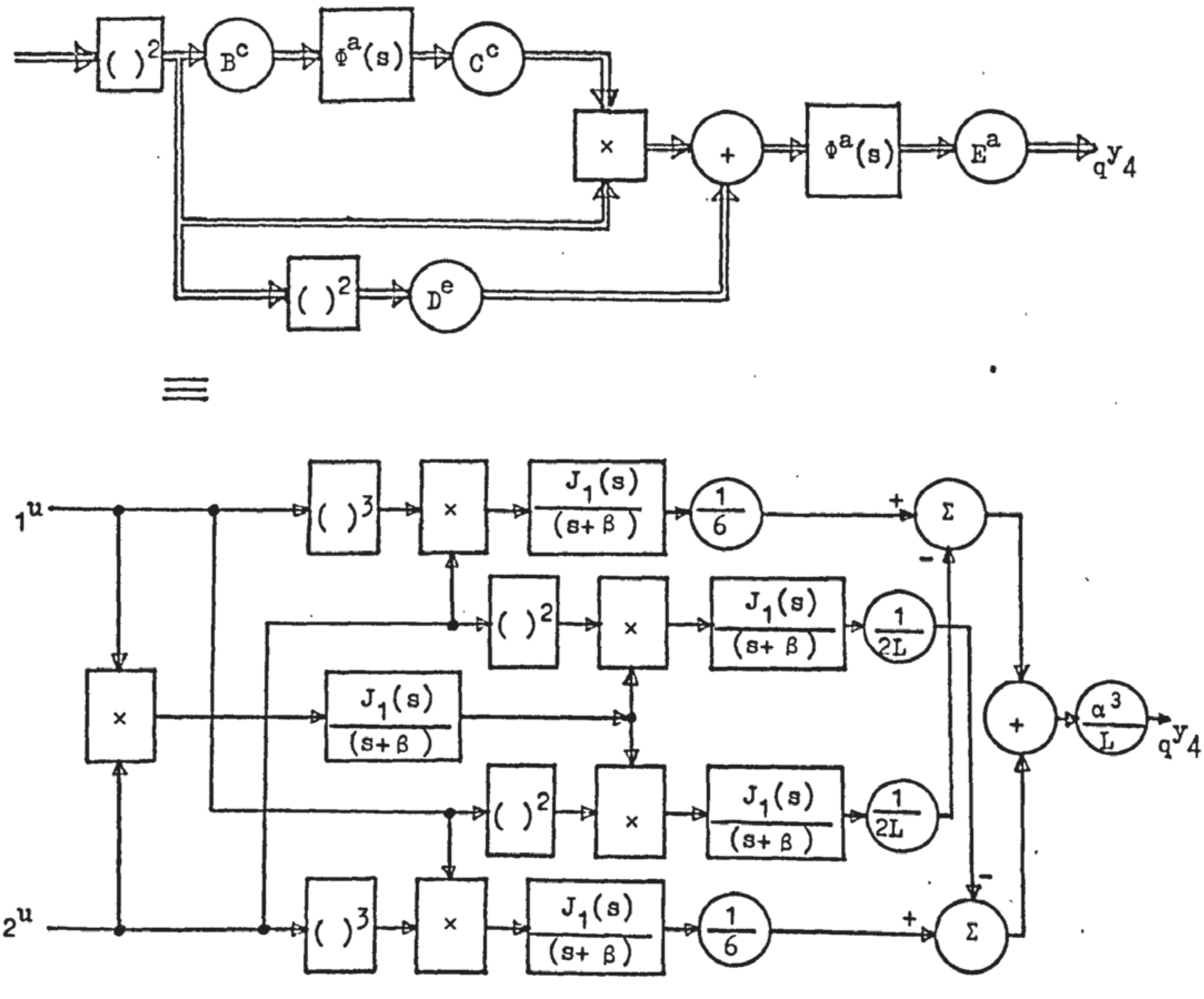


Fig.6.6(b) Fourth-order kernels, $_1W_{41222}$ and $_1W_{41112}$.

CHAPTER 7

STATE VARIABLE DESCRIPTION OF NONLINEAR SYSTEMS-
DISCRETE SYSTEMS

7.1 Introduction

This chapter is mainly concerned with formulation and solution of state variable problem in nonlinear discrete systems characterised by discrete Volterra series. A method is developed to obtain the discrete state equation of a sampled-data nonlinear system from the state transition equation of its continuous part, by characterising the variables at the sampling instants. The solution, in the Volterra series form, of the discrete dynamic equations is then obtained by a procedure developed for nonlinear discrete systems. To illustrate the method, the solution is applied to obtain the sampled response of the direction dependent system of chapter 6. The method is then extended to obtain the state space solution of asynchronous sampled-data systems by means of a multidimensional modified z transformation.

The state space solution of multivariable systems is also investigated in order to obtain the multidimensional z transform kernels of the system. The z transform kernels are then synthesised in terms of the discrete state transition matrix. Finally, the method is illustrated by applying the solution to obtain the discrete response as well as the multidimensional z transform kernels of the direction dependent system of chapter 6. It is assumed here, for simplicity, that the zero-order component in the input signal is equal to zero.

7.2 Representation of Nonlinear Discrete Systems

If the nonlinear system is composed of all discrete elements and is characterised by the discrete Volterra series, then the state variable problem may be solved in a way analogus to the continuous systems. In this section, the method of the previous chapter is extended, using multidimensional z transforms, to obtain the solution of the discrete state and output equations.

7.2.1 Solution of Discrete Dynamic Equations

A single-input, single-output discrete nonlinear system may be described by the following discrete state and output equations.

$$ {}_px(\langle K+1\rangle T) = {}_pA_i\,{}_ix(KT)+{}_pD\,u(KT)+{}_pB_{ij}\,{}_iv(KT)\,{}_jv(KT) + {}_pC_{ijk}\,{}_iv(KT)\,{}_jv(KT)\,{}_kv(KT)+\ldots\ldots \tag{7.2.1} $$

and

$$ y(KT) = E_i\,{}_iv(KT)+F_{ij}\,{}_iv(KT)\,{}_jv(KT)+G_{ijk}\,{}_iv(KT)\,{}_jv(KT)\,{}_kv(KT)+\ldots \tag{7.2.2} $$

where T is the sampling interval. The discrete model representing the state of the system at sampling instants may be obtained by replacing the integrator of the continuous system representation of Fig.6.1, by a delay device and the continuous signals by discrete signals. In the above equations, ${}_px(KT)$ and ${}_pv(KT)$ are state and augmented state vectors, respectively given by

$$ {}_px(KT) = \mathrm{Col}\{{}_1x(KT)\ {}_2x(KT)\ldots\ldots{}_px(KT)\} \quad \text{and} $$
$$ {}_pv(KT) = \mathrm{Col}\{{}_1x(KT)\ {}_2x(KT)\ldots\ldots{}_px(KT)\quad u(KT)\}\ , $$

${}_ix(KT)$, ${}_jx(KT)$, $i=1,2,\ldots.$ and $j=1,2,\ldots.$, etc., are state variables represented by outputs of i^{th} and j^{th} delay devices, respectively, $u(KT)$ is the input variable, all defined at sampling instants only.

The solution to the discrete state equation may be obtained by taking z transform of eqn.(7.2.1) and rearranging, which gives

$$ \left[zI-A\right]{}_pX(z) = z\,{}_px(0)+{}_pD\,U_1(z)+{}_pB_{ij}\,{}_iV(z)\underset{j}{*}V(z)+{}_pC_{ijk}\,{}_iV(z)\underset{j}{*}V(z)\underset{k}{*}V(z)+\ldots \tag{7.2.3} $$

Pre-multiplying both sides by $\Phi(z)=\left[zI-A\right]^{-1}$, gives

$$ {}_pX(z) = {}_p\Phi_i(z)\left[z_ix(0)+{}_iD\,U_1(z)+{}_iB_{jk}\,{}_jV(z)\underset{k}{*}V(z)+{}_iC_{jkl}\,{}_jV(z)\underset{k}{*}V(z)\underset{l}{*}V(z)+\ldots\right] \tag{7.2.4} $$

The solution for ${}_pX(z)$ is sought in the Volterra series form and substituting this solution in eqn.(7.2.4) and equating terms of equal order gives the terms of the Volterra series solution, as

$$ {}_pX_1(z) = {}_p\Phi_i(z) z\, {}_ix(0) + {}_p\Phi_i(z)\, {}_iD\, U_1(z) \tag{7.2.5} $$

$$ {}_pX_2(z_1,z_2) = {}_p\Phi_i(z_1z_2)\, {}_iB_{jk}\, {}_jV_1(z_1)\, {}_kV_1(z_2) \tag{7.2.6} $$

$$ {}_pX_3(z_1,z_2,z_3) = {}_p\Phi_i(z_1z_2z_3)\Big[{}_iB_{jk}\{{}_jV_1(z_1)\,{}_kV_2(z_2,z_3) + {}_jV_2(z_1,z_2)\,{}_kV_1(z_3)\} + {}_iC_{jkl}\, {}_jV_1(z_1)\,{}_kV_1(z_2)\,{}_lV_1(z_3)\Big] \tag{7.2.7} $$

where ${}_ix(0)$ is the initial condition vector and $\phi(KT) = \mathcal{Z}^{-1}[z\,\Phi(z)]$, is the discrete state transition matrix of A defined at sampling instants only. Assuming that the Volterra series solution converges, eqns.(7.2.5) to (7.2.7) provide a solution to the discrete state equation (7.2.1).

The solution to the output equation may be obtained by substituting the Volterra series solutions of Y(z) and ${}_pV(z)$ in the z transform of the output equation(7.2.2) and then equating the terms of equal order, which gives

$$ Y_1(z) = E_i\, {}_iV(z) \tag{7.2.8} $$

$$ Y_2(z_1,z_2) = E_i\, {}_iV_2(z_1,z_2) + F_{ij}\, {}_iV_1(z_1)\, {}_jV_1(z_2) \tag{7.2.9} $$

$$ Y_3(z_1,z_2,z_3) = E_i\, {}_iV_3(z_1,z_2,z_3) + F_{ij}\{{}_iV_1(z_1)\,{}_jV_2(z_2,z_3) + {}_iV_2(z_1,z_2)\,{}_jV_1(z_3)\} + G_{ijk}\, {}_iV_1(z_1)\,{}_jV_1(z_2)\,{}_kV_1(z_3) \tag{7.2.10} $$

etc.

where ${}_iV_1(z) = \left[{}_iX_1(z) \quad U_1(z)\right]$, ${}_iV_2(z_1,z_2) = {}_iX_2(z_1,z_2)$ and ${}_iV_3(z_1,z_2,z_3) = {}_iX_3(z_1,z_2,z_3)$ and are given by eqns.(7.2.5) to (7.2.7). Assuming that the output Volterra series is convergent, eqns.(7.2.8) to (7.2.10) provide a complete state-space solution to the output equation(7.2.2). To obtain the output sequence $y_n(KT)$, the variables $z_1,z_2,\ldots,z_n$ in $Y_n(z_1,z_2,\ldots,z_n)$ are associated, using the association of variables procedure developed in Chapter 2, to obtain $Y_1(z)$ and then using one-dimensional inverse z transformation yields $y_n(KT)$. The above solution clearly shows how the discrete state transition matrix operates on each of the terms of the series to obtain the state and the output of the system in terms of the initial condition vector and the input sequence.

7.2.2 Illustrative Example

The method can be best illustrated by considering a discrete system described by the following nonlinear difference equation, as an example.

$$y_{K+1} = a\, y_K + b\, u_K + c\, y_K^2 + d\, u_K\, y_K \qquad (7.2.11)$$

where u_K is the input sequence and y_K, the output sequence.

Letting ${}_1x_K = y_K$ as the state variable, the dynamic equations may be written as

$$\begin{aligned} {}_pv_{K+1} &= {}_pA_i\ {}_ix_K + {}_pD\ u_K + {}_pB_{1j}\ {}_1v_K\ {}_jv_K \quad ; \quad j = 1,2 \\ y_K &= E_i\ {}_iv_K \end{aligned} \qquad (7.2.12)$$

where $A = [a]$; $D = [b]$; $B = [c \quad d]$ and $E = [1 \quad 0]$. Then,

$$\Phi(z) = (zI - A)^{-1} = \frac{1}{z-a}$$

The solution of the state equation in the form of Volterra series is given by

$$\begin{aligned} {}_1X_1(z) &= {}_1\Phi_1(z)\ z\ {}_1x(0) + {}_1\Phi_i(z)\ {}_iD\ U_1(z) \\ &= \frac{z\ {}_1x(0)}{(z-a)} + \frac{b\ U_1(z)}{(z-a)} \end{aligned} \qquad (7.2.13)$$

$$\begin{aligned} {}_1X_2(z_1,z_2) &= {}_1\Phi_i(z_1 z_2)\,{}_iB_{1j}\ {}_1V_1(z_1)\,{}_jV_1(z_2) \\ &= \frac{1}{(z_1z_2-a)} \begin{bmatrix} c & d \end{bmatrix} \begin{bmatrix} {}_1X_1(z_1)\,{}_1X_1(z_2) \\ {}_1X_1(z_1)\ U_1(z_2) \end{bmatrix} \\ &= \frac{1}{(z_1z_2-a)} \left[\frac{\{{}_1x(0)z_1 + b\ U_1(z_1)\}}{(z_1-a)} \right] \left[\frac{c\{{}_1x(0)z_2 + bU_1(z_2)\}}{(z_2-a)} + d\,U_1(z_2) \right] \end{aligned} \qquad (7.2.14)$$

and

$$\begin{aligned} {}_1X_3(z_1,z_2,z_3) &= {}_1\Phi_i(z_1z_2z_3)\,{}_iB_{1j}\left[{}_1V_1(z_1)\,{}_1V_2(z_2,z_3) + {}_1V_2(z_1,z_2)\,{}_1V_1(z_3)\right] \\ &= \frac{1}{(z_1z_2z_3-a)}\Big[c\{{}_1X_1(z_1)\,{}_1X_2(z_2,z_3) + {}_1X_2(z_1,z_2)\,{}_1X_1(z_3)\} \\ &\qquad + dU_1(z_1)\,{}_1X_2(z_2,z_3)\Big] \end{aligned} \qquad (7.2.15)$$

where ${}_1X_1(z_1)$, ${}_1X_2(z_1,z_2)$ are given by eqns.(7.2.13) and (7.2.14), respectively. The higher order terms may be similarly obtained.

The terms of the Volterra series solution of the output equation are given by

$Y_1(z) = E_i\ {}_iV(z) = {}_1X_1(z)$

$Y_2(z_1,z_2) = E_i\ {}_iV_2(z_1,z_2) + F_{ij}\ {}_iV_1(z_1)_jV_1(z_2) = {}_1X_2(z_1,z_2)$; since $F_{ij} = 0$ and (7.2.16)

$$Y_3(z_1,z_2,z_3) = E_i\ {}_iV_3(z_1,z_2,z_3) + F_{ij}\{{}_iV_1(z_1)_jV_2(z_2,z_3) + {}_iV_2(z_1,z_2)_jV_1(z_3)\} + G_{ijk}\ {}_iV_1(z_1)_jV_1(z_2)_kV_1(z_3)$$

$$= {}_1X_3(z_1,z_2,z_3)\ ;\ \text{since } F_{ij} \text{ and } G_{ijk} \text{ are equal to zero.}$$

where ${}_1X_1(z_1)$, ${}_1X_2(z_1,z_2)$ and ${}_1X_3(z_1,z_2,z_3)$ are given by eqns.(7.2.13) to (7.2.15), respectively. Then, the output sequence y_K may be easily obtained.

7.3 State Variable Representation of Nonlinear Sampled-Data Systems

The sampled-data system, in contrast to the discrete system, consists of a continuous system with sample-and-hold operation applied to its input. This section develops a general method for obtaining the discrete state equation of a sampled-data system from the state transition equation of its continuous part, by characterising the variables at the sampling instants. A complete solution of the discrete state equation is then obtained by a procedure outlined in section 7.2. The method is illustrated by obtaining the discrete response of the direction dependent system of Chapter 6.

7.3.1 Discrete State Equation

The state equation of the continuous part of the system is given by eqn.(6.2.1) and the first three terms of its Volterra series solution are given by

$${}_pV_1(s) = {}_p\Phi_i(s)_iv(0) \tag{7.3.1}$$

$${}_pV_2(s_1,s_2) = {}_p\Phi_i(s_1+s_2)_iB_{jk}\ {}_jV_1(s_1)_kV_1(s_2) \tag{7.3.2}$$

$${}_pV_3(s_1,s_2,s_3) = {}_p\Phi_i(s_1+s_2+s_3)\Big[{}_iB_{jk}\{{}_jV_1(s_1)_kV_2(s_2,s_3) + {}_jV_2(s_1,s_2)_kV_1(s_3)\} + {}_iC_{jkl}\ {}_jV_1(s_1)_kV_1(s_2)_lV_1(s_3)\Big] \tag{7.3.3}$$

This may be written in the form as

$$
\begin{aligned}
{}_pV(s) &= {}_pV_1(s)+{}_pV_2(s_1,s_2)+{}_pV_3(s_1,s_2,s_3)+\ldots\ldots \\
&= {}_p\Phi_i(s)\,{}_iv(0)+{}_p\Theta_{ij}(s_1,s_2)\,{}_iv(0)\,{}_jv(0)+{}_p\Psi_{ijk}(s_1,s_2,s_3)\,{}_iv(0)\,{}_jv(0)\,{}_kv(0) \\
&\quad +\ldots\ldots\ldots \qquad (7.3.4)
\end{aligned}
$$

where $\Phi(s)=(sI-A)^{-1}$; ${}_p\Theta_{ab}(s_1,s_2)={}_p\Phi_i(s_1+s_2)\,{}_iB_{jk}\,{}_j\Phi_a(s_1)\,{}_k\Phi_b(s_2)$

and (7.3.5)

$$
\begin{aligned}
{}_p\Psi_{abc}(s_1,s_2,s_3) = {}_p\Phi_i(s_1+s_2+s_3)\Big[{}_iB_{jk}\{{}_j\Phi_a(s_1)\,{}_k\Theta_{bc}(s_2,s_3) \\
+{}_j\Theta_{ab}(s_1,s_2)\,{}_k\Phi_c(s_3)\}+{}_iC_{jkl}\,{}_j\Phi_a(s_1)\,{}_k\Phi_b(s_2)\,{}_l\Phi_c(s_3)\Big] \qquad (7.3.6)
\end{aligned}
$$

In arriving at the above result, eqns.(7.3.1) to (7.3.3) have been used. Using the real convolution theorem[85], the associated transforms ${}_p\Theta_{ab}(s)$ and ${}_p\Psi_{abc}(s)$ of ${}_p\Theta_{ab}(s_1,s_2)$ and ${}_p\Psi_{abc}(s_1,s_2,s_3)$, respectively, are obtained as

$$
{}_p\Theta_{ab}(s) = {}_p\Phi_i(s)\,{}_iB_{jk}\{{}_j\Phi_a(s) * {}_k\Phi_b(s)\} \qquad (7.3.7)
$$

$$
\begin{aligned}
{}_p\Psi_{abc}(s) = {}_p\Phi_i(s)\Big[{}_iB_{jk}\{{}_j\Phi_a(s) * {}_k\Theta_{bc}(s)+{}_j\Theta_{ab}(s) * {}_k\Phi_c(s)\} \\
+{}_iC_{jkl}\,{}_j\Phi_a(s) * {}_k\Phi_b(s) * {}_l\Phi_c(s)\Big] \qquad (7.3.8)
\end{aligned}
$$

Substituting these equations in eqn.(7.3.4) for ${}_p\Theta_{ij}(s_1,s_2)$ and ${}_p\Psi_{ijk}(s_1,s_2,s_3)$, respectively and inverting gives the state transition equation of the continuous part of the system as

$$
\begin{aligned}
{}_pv(t) = \Big[{}_p\phi_i(t-t_0)\,{}_iv(t_0)+{}_p\theta_{ij}(t-t_0)\,{}_iv(t_0)\,{}_jv(t_0) \\
+{}_p\psi_{ijk}(t-t_0)\,{}_iv(t_0)\,{}_jv(t_0)\,{}_kv(t_0)+\ldots\ldots\Big] \quad ,t \geqslant t_0 \qquad (7.3.9)
\end{aligned}
$$

where, it is to be noted that, the initial time is taken as $t= t_0$.

Letting $t_0= KT$ and $t= (K+1)T$, the discrete state equation characterising the sampled-data system may be obtained as

$$
\begin{aligned}
{}_pv(\langle K+1\rangle T) = \Big[{}_p\phi_i(T)\,{}_iv(KT)+{}_p\theta_{ij}(T)\,{}_iv(KT)\,{}_jv(KT)+{}_p\psi_{ijk}(T)\,{}_iv(KT)\,{}_jv(KT)\,{}_kv(KT) \\
+\ldots\Big] \text{, for } KT \leqslant t \leqslant (K+1)T \qquad (7.3.10)
\end{aligned}
$$

where ${}_p\phi_a(T) = L^{-1}\{{}_p\Phi_a(s)\}_{t=T}$,

$$
{}_p\theta_{ab}(T) = \int_0^T {}_p\phi_i(T-\lambda)\,{}_iB_{jk}\,{}_j\phi_a(\lambda)\,{}_k\phi_b(\lambda)\,d\lambda \qquad (7.3.11)
$$

$$ {}_p\psi_{abc}(T) = \int_0^T {}_p\phi_i(T-\lambda)\Big[{}_iB_{jk}\{{}_j\phi_a(\lambda)_k\theta_{bc}(\lambda)+{}_j\theta_{ab}(\lambda)_k\phi_c(\lambda)\} + {}_iC_{jkl}\;{}_j\phi_a(\lambda)_k\phi_b(\lambda)_l\phi_c(\lambda)\Big]\,d\lambda $$

Next, the solution of the discrete state equation is obtained.

7.3.2 Method of Solution

The discrete state eqn.(7.3.10), describing the sampled-data non-linear system at sampling instants, may be solved by the procedure outlined in section 7.2. Then, the terms of the Volterra series solution of ${}_pV(z)$ are given by

$$ {}_pV_1(z) = {}_p\Phi_i(z)\; z\; {}_iv(0) \tag{7.3.12} $$

$$ {}_pV_2(z_1,z_2) = {}_p\Phi_i(z_1z_2)_i\theta_{jk}(T)_jV_1(z_1)_kV_1(z_2) \tag{7.3.13} $$

$$ {}_pV_3(z_1,z_2,z_3) = {}_p\Phi_i(z_1.z_2.z_3)\Big[{}_i\theta_{jk}(T)\{{}_jV_1(z_1)_kV_2(z_2,z_3)+{}_jV_2(z_1,z_2)_kV_1(z_3)\} + {}_i\psi_{jkl}(T)_jV_1(z_1)_kV_1(z_2)_lV_1(z_3)\Big] \tag{7.3.14} $$

where ${}_iv(0)$ is the initial condition vector, $\Phi(z)=[zI-\phi(T)]^{-1}$, ${}_p\phi_a(T)$, ${}_p\theta_{ab}(T)$ and ${}_p\psi_{abc}(T)$ are given by eqns.(7.3.11).

The discrete state transition equation of the sampled-data system can be obtained, in the same way as the state transition equation of the continuous system, as

$$ {}_pv(KT) = {}_p\phi_i(KT)_iv(0)+{}_p\delta_{ij}(KT)_iv(0)_jv(0)+{}_p\gamma_{ijk}(KT)_iv(0)_jv(0)_kv(0)+\ldots\ldots \tag{7.3.15} $$

where

$$ {}_p\delta_{ab}(KT) = \sum_{l=0}^{K-1} {}_p\phi_I(\langle K-l-1\rangle T)_I\theta_{JK}(T)_J\phi_a(lT)_K\phi_b(lT) \quad , $$

$$ {}_p\gamma_{abc}(KT) = \sum_{l=0}^{K-1} {}_p\phi_I(\langle K-l-1\rangle T)\Big[{}_I\theta_{JK}(T)\{{}_J\phi_a(lT)_K\delta_{bc}(lT)+{}_J\delta_{ab}(lT)_K\phi_c(lT)\} + {}_I\psi_{JKw}(T)_J\phi_a(lT)_K\phi_b(lT)_w\phi_c(lT)\Big] \tag{7.3.16} $$

It may be observed that ${}_p\delta_{ij}(KT)$ and ${}_p\gamma_{ijk}(KT)$ are inverse z transforms of ${}_p\Delta_{ij}(z)$ and ${}_p\Gamma_{ijk}(z)$, respectively, where ${}_p\Delta_{ij}(z)$ and ${}_p\Gamma_{ijk}(z)$ are associated z transforms of ${}_p\Delta_{ij}(z_1,z_2)$ and ${}_p\Gamma_{ijk}(z_1,z_2,z_3)$, respectively, obtained, using the complex convolution theorem and the real discrete convolution theorem of Chapter 3, as

$$
\begin{aligned}
{}_p\Delta_{ab}(z) &= {}_p\Phi_I(z)\,{}_I\theta_{JK}(T)\,{}_J\Phi_a(z)z * {}_K\Phi_b(z)z \\
{}_p\Gamma_{abc}(z) &= {}_p\Phi_I(z)\Big[{}_I\theta_{JK}(T)\{{}_J\Phi_a(z)z * {}_K\Delta_{bc}(z) + {}_J\Delta_{ab}(z) * {}_K\Phi_c(z)z\} \\
&\qquad + {}_I\psi_{JKW}(T)\,{}_J\Phi_a(z)z * {}_K\Phi_b(z)z * {}_W\Phi_c(z)z\Big]
\end{aligned}
\tag{7.3.17}
$$

It is interesting to note that the discrete state transition equation(7.3.15) is analogous to its continuous couterpart in eqn.(7.3.9). The state transition equation of the continuous nonlinear system describes the state of the system for $t \geq t_0$ for any input defined over the same interval. The discrete state transition equation of the digital system, on the other hand, is more restrictive in that it describes the state only at the sampling instants $t = KT$, $K = 0,1,2,\ldots\ldots$ and only for a system whose inputs are applied through a zero-order hold.

The output equation describing the measurement process, of the sampled-data system, at sampling instants is given by eqn.(7.2.2). Thus, the terms of the Volterra series solution of the output equation are given by eqns.(7.2.8) to (7.2.10), where ${}_pV_1(z_1)$, ${}_pV_2(z_1,z_2)$, etc., are given by eqns.(7.3.12) to (7.3.14). This method of solution has a restriction in that it requires the type of the input to be specified. Thus, it may not be possible with this method to obtain the relationship between the Volterra kernels and the transition matrix of the system. However, a more general method of solution is developed for multivariable systems in section 7.5, which not only allows the relationship between the Volterra kernels and the transition matrix of the system to be established, but also provides an explicit input-output expression.

7.3.3 Example - Direction Dependent System

The method is illustrated by considering the direction dependent system of Chapter 5 with sample-and-hold operation applied to its input as shown in Fig.7.1. The state equation of the continuous system is given, in the matrix form, by

$$
\begin{bmatrix} {}_1\dot{x} \\ \dot{u} \end{bmatrix} = \underbrace{\begin{bmatrix} -\omega & \omega \\ 0 & 0 \end{bmatrix}}_{t_A} \begin{bmatrix} {}_1x \\ u \end{bmatrix} + \underbrace{\begin{bmatrix} -r & r \\ 0 & 0 \end{bmatrix}}_{t_B} \begin{bmatrix} u & {}_1x \\ u & u \end{bmatrix}
\tag{7.3.18}
$$

where ${}_1x = y$ is the state variable characterising the continuous part of the system. Eqn.(7.3.18) may be written as

$$ {}_p\dot{v} = {}_pA_i\ {}_iv + {}_pB_{ij}\ {}_iv\ {}_jv \quad ; \quad i,j = 1, (P+1); \tag{7.3.19}$$

For the continuous part of the system, the Laplace transform of the state transition matrix is given by

$$\phi(s) = \begin{bmatrix} \frac{1}{s+\omega} & \frac{\omega}{s(s+\omega)} \\ 0 & \frac{1}{s} \end{bmatrix} \tag{7.3.20}$$

The discrete state transition matrix in z transform is obtained as

$$z\ \phi(z) = z\left[zI - \phi(T)\right]^{-1} = \begin{bmatrix} \frac{z}{(z-e^{-\omega T})} & \frac{z(1-e^{-\omega T})}{(z-1)(z-e^{-\omega T})} \\ 0 & \frac{z}{(z-1)} \end{bmatrix} \tag{7.3.21}$$

where, it is to be noted that, $\phi(t)$ is the inverse Laplace transform of eqn.(7.3.20), given by

$$\phi(T) = \begin{bmatrix} e^{-\omega T} & (1-e^{-\omega T}) \\ 0 & 1 \end{bmatrix} \tag{7.3.22}$$

Next, the solution of discrete state equation is obtained.

<u>First order term</u>

The linear term of the Volterra series solution of the discrete state equation is given by

$$ {}_pV_1(z) = {}_p\phi_I(z)\ z\ {}_Iv(0)$$

$$\text{Therefore,}\quad {}_1X_1(z) = \left\{\frac{z\ {}_1x(0)}{(z-e^{-\omega T})} + \frac{z(1-e^{-\omega T})}{(z-1)(z-e^{-\omega T})}\right\} \tag{7.3.23}$$

<u>Second order term</u>: From eqns.(7.3.11), ${}_1\theta_{(P+1)1}(T)$ and ${}_1\theta_{(P+1)(P+1)}(T)$ are obtained as

$$ {}_1\theta_{(P+1)1}(T) = -r\int_0^T {}_1\phi_1(T-\tau)\ {}_{P+1}\phi_{P+1}(\tau)\ {}_1\phi_1(\tau)\,d\tau$$

$$= -r\int_0^T e^{-\omega T}\ d\tau = -rT\ e^{-\omega T}.$$

$$ {}_1\theta_{(P+1)(P+1)}(T) = \int_0^T {}_1\phi_1(T-\tau)\left[-r\ {}_{P+1}\phi_{P+1}(\tau)\ {}_1\phi_{P+1}(\tau) + r\ {}_{P+1}\phi_{P+1}(\tau)\ {}_{P+1}\phi_{P+1}(\tau)\right]d\tau$$

$$= \int_0^T e^{-\omega(T-\tau)}\{-r(1-e^{-\omega\tau})+r\}d\tau = rT\,e^{-\omega T}$$

The second order term is then given by

$$ {}_1X_2(z_1,z_2) = {}_1\phi_1(z_1z_2)\Big[{}_1\theta_{(P+1)1}(T)\,{}_{P+1}V_1(z_1)\,{}_1V_1(z_2) + {}_1\theta_{(P+1)(P+1)}(T)\,{}_{P+1}V_1(z_1)\,{}_{P+1}V_1(z_2)\Big]$$

$$= \frac{1}{(z_1z_2-e^{-\omega T})}\{-rT\,e^{-\omega T}\,U_1(z_1)\,{}_1X_1(z_2)+rT\,e^{-\omega T}\,U_1(z_1)U_1(z_2)\}$$

$$= \frac{rT\,e^{-\omega T}\,z_1z_2}{(z_1-1)(z_2-e^{-\omega T})(z_1z_2-e^{-\omega T})} - \frac{rT\,e^{-\omega T}\,z_1z_2\;{}_1x(0)}{(z_1-1)(z_2-e^{-\omega T})(z_1z_2-e^{-\omega T})} \tag{7.3.24}$$

where $U_1(z)=\frac{z}{(z-1)}$, since $u(KT)$ is a sampled step input.

<u>Third order term</u>: From eqn.(7.3.11), ${}_1\psi_{(P+1)(P+1)1}(T)$ and ${}_1\psi_{(P+1)(P+1)(P+1)}(T)$ are obtained as

$${}_1\psi_{(P+1)(P+1)1}(T) = -r\int_0^T {}_1\phi_1(T-\tau)\,{}_{P+1}\phi_{P+1}(\tau)\;{}_1\theta_{(P+1)1}(\tau)\,d\tau$$

$$= r^2\int_0^T \tau\,e^{-\omega T}\,d\tau = \frac{r^2T^2}{2}\,e^{-\omega T}$$

$${}_1\psi_{(P+1)(P+1)(P+1)}(T) = -r\int_0^T {}_1\phi_1(T-\tau)\,{}_{P+1}\phi_{P+1}(\tau)\,{}_1\theta_{(P+1)(P+1)}(\tau)\,d\tau$$

$$= -r^2\,e^{-\omega T}\int_0^T \tau\,d\tau = -\frac{r^2T^2}{2}\,e^{-\omega T}$$

Then, the third order term is given by

$$ {}_1X_3(z_1,z_2,z_3) = {}_1\phi_1(z_1z_2z_3)\Big[{}_1\theta_{(P+1)1}(T)\,{}_{P+1}V_1(z_1)\,{}_1V_2(z_2,z_3) + {}_1\psi_{(P+1)(P+1)1}(T)\,{}_{P+1}V_1(z_1)\,{}_{P+1}V_1(z_2)\,{}_1V_1(z_3) + {}_1\psi_{(P+1)(P+1)(P+1)}(T)\,{}_{P+1}V_1(z_1)\,{}_{P+1}V_1(z_2)\,{}_{P+1}V_1(z_3)\Big].$$

$$= \frac{1}{(z_1z_2z_3-e^{-\omega T})}\{{}_1\theta_{(P+1)1}(T)U_1(z_1)\,{}_1X_2(z_2,z_3) + {}_1\psi_{(P+1)(P+1)1}(T)U_1(z_1)U_1(z_2)\,{}_1X_1(z_3) + {}_1\psi_{(P+1)(P+1)(P+1)}(T)U_1(z_1)U_1(z_2)U_1(z_3)\}$$

$$= -\frac{r^2T^2\, e^{-\omega T}\, z_1z_2z_3(z_2z_3+e^{-\omega T})\{1-{}_1x(0)\}}{2(z_1-1)(z_2-1)(z_3-e^{-\omega T})(z_2z_3-e^{-\omega T})(z_1z_2z_3-e^{-\omega T})} \qquad (7.3.25)$$

The first three terms of the Volterra series solution of the discrete output equation are given by

$$Y_1(z) = {}_1X_1(z)\ , \qquad\qquad Y_2(z_1,z_2) = {}_1X_2(z_1,z_2)$$

and $\quad Y_3(z_1,z_2,z_3) = {}_1X_3(z_1,z_2,z_3)$

where ${}_1X_1(z)$, ${}_1X_2(z_1,z_2)$ and ${}_1X_3(z_1,z_2,z_3)$ are given by eqns.(7.3.23) to (7.3.25), respectively, and $y(0)={}_1x(0)$. It is interesting to note that, when the initial condition ${}_1x(0)$ is equal to zero, the above results agree exactly with those obtained in Chapter 5 for direction dependent system.

7.4 Asynchronous Sampled-Data Systems

When the input-output sampling is not synchronous but have the same period, then the state space problem of the system may be solved in a way similar to the synchronous sampled case of section 7.3, but using multi-dimensional modified z transforms. To obtain the response between the sampling instants, it is necessary to put $t_0=(K+m)T$ and $t=(K+1+m)T$ in the state transition equation(7.3.9) of the continuous part, which yields the discrete state equation of the asynchronous sampled-data system as

$$\begin{aligned} {}_pv(\langle K+1+m\rangle T) &= {}_p\phi_i(T)\,{}_iv(\langle K+m\rangle T)+{}_p\theta_{ij}(T)\,{}_iv(\langle K+m\rangle T)\,{}_jv(\langle K+m\rangle T) \\ &\quad +{}_p\psi_{ijk}(T)\,{}_iv(\langle K+m\rangle T)\,{}_jv(\langle K+m\rangle T)\,{}_kv(\langle K+m\rangle T) + \ldots. \end{aligned} \qquad (7.4.1)$$

where m is a dummy variable, $0\leqslant m<1$, and ${}_p\phi_a(T)$, ${}_p\theta_{ab}(T)$ and ${}_p\psi_{abc}(T)$ are given by eqns.(7.3.11).

The output equation, to represent the response between the sampling instants, may be similarly obtained as

$$\begin{aligned} y(\langle K+m\rangle T) &= E_i\ {}_iv(\langle K+m\rangle T)+ F_{ij}\ {}_iv(\langle K+m\rangle T)\,{}_jv(\langle K+m\rangle T) \\ &\quad + G_{ijk}\ {}_iv(\langle K+m\rangle T)\,{}_jv(\langle K+m\rangle T)\,{}_kv(\langle K+m\rangle T) + \ldots. \end{aligned} \qquad (7.4.2)$$

The solution to the discrete state equation may be obtained in the

Volterra series form, using the procedure developed in section 7.2, as

$$ {}_pV_1(m,z) = {}_p\Phi_i(z)\, z\; {}_iv(mT) \tag{7.4.3}$$

$$ {}_pV_2(m,z_1,z_2) = {}_p\Phi_i(z_1z_2)\, {}_i\theta_{jk}(T)\, {}_jV_1(m,z_1)\, {}_kV_1(m,z_2) \tag{7.4.4}$$

$$ \begin{aligned} {}_pV_3(m,z_1,z_2,z_3) = {}_p\Phi_i(z_1z_2z_3)\Big[& {}_i\theta_{jk}(T)\{ {}_jV_1(m,z_1)\, {}_kV_2(m,z_2,z_3) \\ & + {}_jV_2(m,z_1,z_2)\, {}_kV_1(m,z_3)\} + {}_i\psi_{jkl}(T)\, {}_jV_1(m,z_1)\, {}_kV_1(m,z_2)\, {}_lV_1(m,z_3)\Big] \end{aligned}$$

$$ \text{etc.,} \quad 0 \leq m < 1 \tag{7.4.5}$$

where $v(mT)$ is given by

$$ {}_pv(mT) = \Big[{}_p\phi_i(mT)\, {}_iv(0) + {}_p\theta_{ij}(mT)\, {}_iv(0)\, {}_jv(0) + {}_p\psi_{ijk}(mT)\, {}_iv(0)\, {}_jv(0)\, {}_kv(0) \Big], \quad 0 \leq m < 1 \tag{7.4.6}$$

where ${}_p\phi_a(mT)$, ${}_p\theta_{ab}(mT)$ and ${}_p\psi_{abc}(mT)$ may be obtained from eqns.(7.3.11) by replacing the upper limit of integration by mT. It should be noted that the above equation is obtained by putting $t_0=0$ and $t=mT$ in the state transition equation (7.3.9) of the continuous part.

Following the procedure outlined in section 7.2, the terms of the Volterra series solution of the output equation (7.4.2) may be obtained as

$$ Y_1(m,z) = E_i\; {}_iV_1(m,z) \tag{7.4.7}$$

$$ Y_2(m,z_1,z_2) = E_i\; {}_iV_2(m,z_1,z_2) + F_{ij}\; {}_iV_1(m,z_1)\, {}_jV_1(m,z_2) \tag{7.4.8}$$

$$ \begin{aligned} Y_3(m,z_1,z_2,z_3) = \Big[& E_i\; {}_iV_3(m,z_1,z_2,z_3) + F_{ij}\{ {}_iV_1(m,z_1)\, {}_jV_2(m,z_2,z_3) \\ & + {}_iV_2(m,z_1,z_2)\, {}_jV_1(m,z_3)\} + G_{ijk}\; {}_iV_1(m,z_1)\, {}_jV_1(m,z_2)\, {}_kV_1(m,z_3)\Big] \end{aligned}$$

$$ 0 \leq m < 1 \tag{7.4.9}$$

where ${}_pV_1(m,z)$, ${}_pV_2(m,z_1,z_2)$, etc., are given by eqns.(7.4.3) to (7.4.5). By using eqns.(7.4.3) to (7.4.9), the response of the sampled-data system between the sampling instants may be obtained, for $0 \leq m < 1$.

7.5 Multivariable Nonlinear Sampled-Data Systems

An R-input, Q-output sampled-data nonlinear system is obtained when sample-and-hold operation is applied to the inputs of an R-input, Q-output continuous nonlinear system characterised by the generalised dynamic equations (6.3.1) and (6.3.2) and is shown in Fig.7.2. This system has

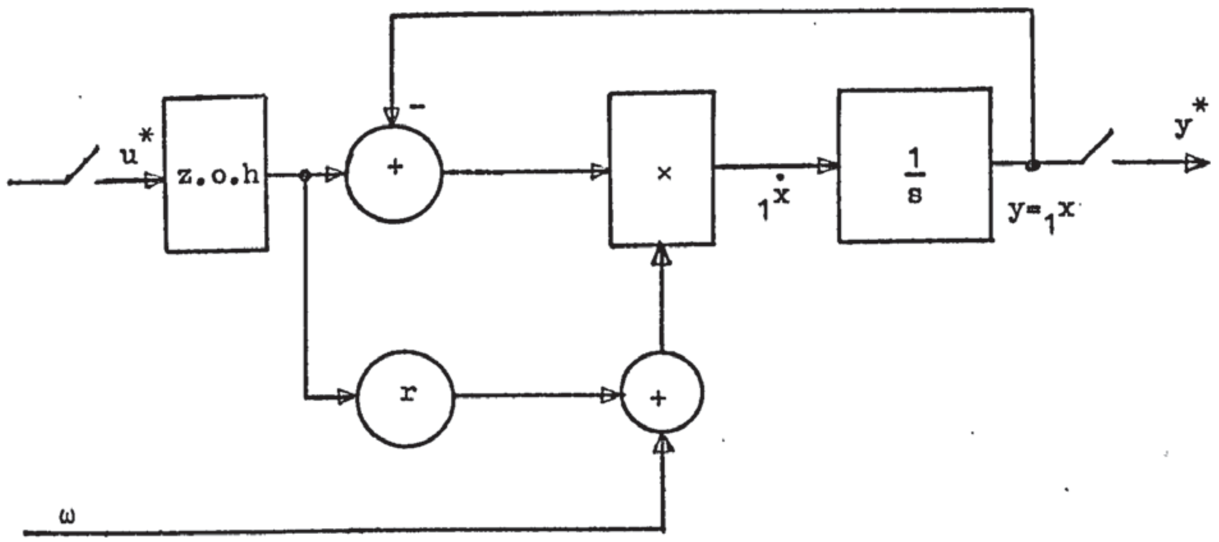


Fig.7.1 Direction dependent system with input applied through sample-and-hold.

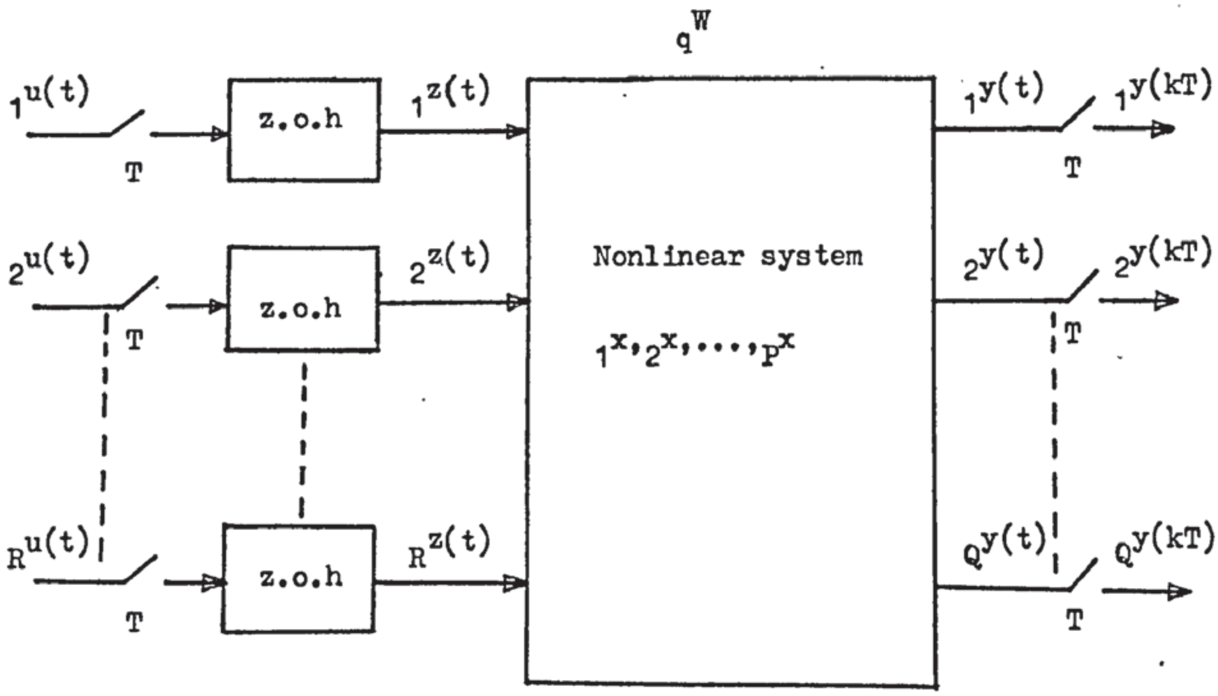


Fig.7.2 An R-input, Q-output nonlinear sampled-data system, with inputs applied through zero-order hold, characterised in state space.

an enormous practical significance, since it enables the continuous nonlinear system to be simulated on digital computer. The discrete dynamic equations characterising the system in Fig.7.2 are derived from the state transition equation of the corresponding continuous system and their solutions obtained. From the solution of the output equation, the multidimensional z transform kernel matrices are derived and a procedure for their synthesis, in terms of discrete state transition matrix, is given.

7.5.1 State Space Characterisation of Sampled-Data Multivariable Systems

The discrete state equation characterising the state of the system at the sampling instants can be obtained from the state transition equation of its continuous part. The state equation of the continuous system is given by eqn.(6.3.1) and the first three terms of its Volterra series solution are given by eqns.(6.3.4) to (6.3.6), where $\Phi^{a}(s)$ is given by

$$\Phi^{a}(s) = \left[sI - A^{a}\right]^{-1} \tag{7.5.1}$$

Substituting eqns.(6.3.4) to (6.3.6) into the Volterra series expansion of ${}_{p}X(s)$ and following the same procedure as in section 7.3, the discrete state equation characterising the multivariable nonlinear sampled-data system may be obtained(see Appendix A.7, for derivation) as

$$\begin{aligned}
{}_{p}x(\langle K+1\rangle T) = {} & \left[{}_{p}\phi^{a}_{i}(T) \;\middle|\; {}_{p}\phi^{b}_{L}(T) \right] \left[\begin{array}{c} {}_{i}x(KT) \\ \hline {}_{L}u(KT) \end{array} \right] \\
& + \left[{}_{p}\theta^{a}_{ij}(T) \;\middle|\; {}_{p}\theta^{b}_{iL}(T) \;\middle|\; {}_{p}\theta^{c}_{LM}(T) \right] \left[\begin{array}{c} {}_{i}x(KT)\,{}_{j}x(KT) \\ \hline {}_{i}x(KT)\,{}_{L}u(KT) \\ \hline {}_{L}u(KT)\,{}_{M}u(KT) \end{array} \right] \\
& + \left[{}_{p}\psi^{a}_{ijk}(T) \;\middle|\; {}_{p}\psi^{b}_{ijL}(T) \;\middle|\; {}_{p}\psi^{c}_{iLM}(T) \;\middle|\; {}_{p}\psi^{d}_{LMN}(T) \right] \left[\begin{array}{c} {}_{i}x(KT)\,{}_{j}x(KT)\,{}_{k}x(KT) \\ \hline {}_{i}x(KT)\,{}_{j}x(KT)\,{}_{L}u(KT) \\ \hline {}_{i}x(KT)\,{}_{L}u(KT)\,{}_{M}u(KT) \\ \hline {}_{L}u(KT)\,{}_{M}u(KT)\,{}_{N}u(KT) \end{array} \right] \\
& + \ldots\ldots\quad \text{for } KT \leqslant t \leqslant (K+1)T
\end{aligned} \tag{7.5.2}$$

where ${}_p\phi^a_i(T) = L^{-1}\{{}_p\Phi^a_i(s)\}_{t=T}$,

$${}_p\phi^b_L(T) = \int_0^T {}_p\phi^a_J(T-\tau)\,{}_J A^b_L\, d\tau \quad ,$$

$${}_p\theta^a_{ij}(T) = \int_0^T {}_p\phi^a_I(T-\tau)\,{}_I B^a_{JK}\;{}_J\phi^a_i(\tau)\,{}_K\phi^a_j(\tau)\,d\tau \quad ,$$

$${}_p\theta^b_{iL}(T) = \int_0^T {}_p\phi^a_I(T-\tau)\Big[{}_I B^a_{JK}\{{}_J\phi^a_i(\tau)\,{}_K\phi^b_L(\tau)+{}_J\phi^b_L(\tau)\,{}_K\phi^a_i(\tau)\}+{}_I B^b_{jL}\;{}_j\phi^a_i(\tau)\Big]\,d\tau \,,$$

$${}_p\theta^c_{LM}(T) = \int_0^T {}_p\phi^a_I(T-\tau)\Big[{}_I B^a_{ij}\;{}_i\phi^b_L(\tau)\,{}_j\phi^b_M(\tau)+{}_I B^b_{iL}\;{}_i\phi^b_M(\tau)+{}_I B^c_{LM}\Big]\,d\tau \quad ,$$

$$\begin{aligned}{}_p\psi^a_{ijk}(T) = \int_0^T {}_p\phi^a_I(T-\tau)\Big[&{}_I B^a_{ST}\{{}_S\phi^a_i(\tau)\,{}_T\theta^a_{jk}(\tau)+{}_S\theta^a_{ij}(\tau)\,{}_T\phi^a_k(\tau)\}\\ &+{}_I C^a_{STZ}\;{}_S\phi^a_i(\tau)\,{}_T\phi^a_j(\tau)\,{}_Z\phi^a_k(\tau)\Big]\,d\tau \quad ,\end{aligned}$$

$$\begin{aligned}{}_p\psi^b_{ijL}(T) = \int_0^T {}_p\phi^a_I(T-\tau)\Big[&{}_I B^a_{ST}\{{}_S\phi^a_i(\tau)\,{}_T\theta^b_{jL}(\tau)+{}_S\phi^b_L(\tau)\,{}_T\theta^a_{ij}(\tau)+{}_S\theta^a_{ij}(\tau)\,{}_T\phi^b_L(\tau)\\ &+{}_S\theta^b_{iL}(\tau)\,{}_T\phi^a_j(\tau)\}+{}_I C^a_{STZ}\{{}_S\phi^a_i(\tau)\,{}_T\phi^a_j(\tau)\,{}_Z\phi^b_L(\tau)+{}_S\phi^a_i(\tau)\,{}_Z\phi^a_j(\tau)\,{}_T\phi^b_L(\tau)\\ &+{}_S\phi^b_L(\tau)\,{}_T\phi^a_i(\tau)\,{}_Z\phi^a_j(\tau)\}+{}_I B^b_{SL}\;{}_S\theta^a_{ij}(\tau)+{}_I C^b_{STL}\;{}_S\phi^a_i(\tau)\,{}_T\phi^a_j(\tau)\Big]\,d\tau \,,\end{aligned}$$

$$\begin{aligned}{}_p\psi^c_{iLM}(T) = \int_0^T {}_p\phi^a_I(T-\tau)\Big[&{}_I B^a_{ST}\{{}_S\phi^a_i(\tau)\,{}_T\theta^c_{LM}(\tau)+{}_S\theta^b_{iL}(\tau)\,{}_T\phi^b_M(\tau)+{}_S\phi^b_L(\tau)\,{}_T\theta^b_{iM}(\tau)\\ &+{}_S\theta^c_{LM}(\tau)\,{}_T\phi^a_i(\tau)\}+{}_I C^a_{STZ}\{{}_S\phi^a_i(\tau)\,{}_T\phi^b_L(\tau)\,{}_Z\phi^b_M(\tau)+{}_S\phi^b_L(\tau)\,{}_T\phi^a_i(\tau)\,{}_Z\phi^b_M(\tau)\\ &+{}_S\phi^b_L(\tau)\,{}_T\phi^b_M(\tau)\,{}_Z\phi^a_i(\tau)\}+{}_I C^b_{STL}\{{}_S\phi^a_i(\tau)\,{}_T\phi^b_M(\tau)+{}_S\phi^b_M(\tau)\,{}_T\phi^a_i(\tau)\}\\ &+{}_I B^b_{SM}\;{}_S\theta^b_{iL}(\tau)+{}_I C^c_{SLM}\;{}_S\phi^a_i(\tau)\Big]\,d\tau \quad ,\end{aligned}$$

and

$$\begin{aligned}{}_p\psi^d_{LMN}(T) = \int_0^T {}_p\phi^a_I(T-\tau)\Big[&{}_I B^a_{ST}\{{}_S\phi^b_L(\tau)\,{}_T\theta^c_{MN}(\tau)+{}_S\theta^c_{LM}(\tau)\,{}_T\phi^b_N(\tau)\}+{}_I B^b_{SN}\;{}_S\theta^c_{LM}(\tau)\\ &+{}_I C^a_{STZ}\;{}_S\phi^b_L(\tau)\,{}_T\phi^b_M(\tau)\,{}_Z\phi^b_N(\tau)+{}_I C^b_{STN}\;{}_S\phi^b_L(\tau)\,{}_T\phi^b_M(\tau)\\ &+{}_I C^c_{SMN}\;{}_S\phi^b_L(\tau)+{}_I C^d_{LMN}\Big]\,d\tau \qquad (7.5.3)\end{aligned}$$

It is to be noted that, in the above derivation, the input vector ${}_L z(t) = {}_L u(KT)$, is taken as a constant vector over the sampling interval, since the signals ${}_L z(t)$, $L = 1,2,\ldots,R$ are the outputs of the zero-order hold.

The output equation of the sampled-data system describing the response at the sampling instants is given by the discrete version of eqn.(6.3.2), which may be written as

$$
{}_q y(KT) = \left[{}_q E^a_i \;\vdots\; {}_q E^b_L\right]\begin{bmatrix} {}_i x(KT) \\ \hdashline {}_L u(KT)\end{bmatrix} + \left[{}_q F^a_{ij} \;\vdots\; {}_q F^b_{iL} \;\vdots\; {}_q F^c_{LM}\right]\begin{bmatrix} {}_i x(KT)\,{}_j x(KT) \\ \hdashline {}_i x(KT)\,{}_L u(KT) \\ \hdashline {}_L u(KT)\,{}_M u(KT)\end{bmatrix}
$$

$$
+ \left[{}_q G^a_{ijk} \;\vdots\; {}_q G^b_{ijL} \;\vdots\; {}_q G^c_{iLM} \;\vdots\; {}_q G^d_{LMN}\right]\begin{bmatrix} {}_i x(KT)\,{}_j x(KT)\,{}_k x(KT) \\ \hdashline {}_i x(KT)\,{}_j x(KT)\,{}_L u(KT) \\ \hdashline {}_i x(KT)\,{}_L u(KT)\,{}_M u(KT) \\ \hdashline {}_L u(KT)\,{}_M u(KT)\,{}_N u(KT)\end{bmatrix}
$$

$$
+ \cdots\cdots \tag{7.5.4}
$$

The discrete model representing the state and the output of the system at the sampling instants $t=KT$, is same as the analogue model shown in Fig.6.3 except that the integrator is replaced by a delay device and the coefficients A^a, A^b, B^a, B^b, B^c, C^a, C^b, C^c and C^d are replaced by the coefficients $\phi^a(T)$, $\phi^b(T)$, $\theta^a(T)$, $\theta^b(T)$, $\theta^c(T)$, $\psi^a(T)$, $\psi^b(T)$, $\psi^c(T)$ and $\psi^d(T)$, respectively.

7.5.2 Solution of Discrete Dynamic Equations

The solution of the above discrete dynamic equations can be obtained in the same way as in the continuous case and hence will not be repeated here. However, the terms of the Volterra series expansion of ${}_p X(z)$ are

$$
{}_p X_1(z) = \{{}_p \phi^a_I(z)\; z\; {}_I x(0) + {}_p \phi^a_I(z)\,{}_I \phi^b_L(T)\,{}_L U_1(z)\} \tag{7.5.5}
$$

$$
{}_p X_2(z_1,z_2) = {}_p \phi^a_I(z_1 z_2)\left[{}_I \theta^a_{ij}(T) \;\vdots\; {}_I \theta^b_{iL}(T) \;\vdots\; {}_I \theta^c_{LM}(T)\right]\begin{bmatrix} {}_i X_1(z_1)\,{}_j X_1(z_2) \\ \hdashline {}_i X_1(z_1)\,{}_L U_1(z_2) \\ \hdashline {}_L U_1(z_1)\,{}_M U_1(z_2)\end{bmatrix} \tag{7.5.6}
$$

$$
\begin{aligned}
{}_p X_3(z_1,z_2,z_3) = {}_p \phi^a_I(z_1 z_2 z_3)\Big\{ & \left[{}_I \theta^a_{ij}(T) \;\vdots\; {}_I \theta^b_{iL}(T)\right]\begin{bmatrix} {}_i X_1(z_1)\,{}_j X_2(z_2,z_3) + {}_i X_2(z_1,z_2)\,{}_j X_1(z_3) \\ \hdashline {}_i X_2(z_1,z_2)\,{}_L U_1(z_3)\end{bmatrix} \\
& + \left[{}_I \psi^a_{ijk}(T) \;\vdots\; {}_I \psi^b_{ijL}(T) \;\vdots\; {}_I \psi^c_{iLM}(T) \;\vdots\; {}_I \psi^d_{LMN}(T)\right]\begin{bmatrix} {}_i X_1(z_1)\,{}_j X_1(z_2)\,{}_k X_1(z_3) \\ \hdashline {}_i X_1(z_1)\,{}_j X_1(z_2)\,{}_L U_1(z_3) \\ \hdashline {}_i X_1(z_1)\,{}_L U_1(z_2)\,{}_M U_1(z_3) \\ \hdashline {}_L U_1(z_1)\,{}_M U_1(z_2)\,{}_N U_1(z_3)\end{bmatrix}\Big\}
\end{aligned} \tag{7.5.7}
$$

Similarly, the terms of the Volterra series solution of the discrete output equation (7.5.4) are given by

$$
{}_qY_1(z) = {}_qE^a_i\,{}_iX_1(z) + {}_qE^b_L\,{}_LU_1(z) \tag{7.5.8}
$$

$$
{}_qY_2(z_1,z_2) = {}_qE^a_i\,{}_iX_2(z_1,z_2) + \left[{}_qF^a_{ij} \,\vdots\, {}_qF^b_{iL} \,\vdots\, {}_qF^c_{LM}\right]\left[\begin{array}{c} {}_iX_1(z_1)\,{}_jX_1(z_2) \\ \hdashline {}_iX_1(z_1)\,{}_LU_1(z_2) \\ \hdashline {}_LU_1(z_1)\,{}_MU_1(z_2) \end{array}\right] \tag{7.5.9}
$$

$$
\begin{aligned}
{}_qY_3(z_1,z_2,z_3) = {}&{}_qE^a_i\,{}_iX_3(z_1,z_2,z_3) + \left[{}_qF^a_{ij} \,\vdots\, {}_qF^b_{iL}\right]\left[\begin{array}{c} {}_iX_1(z_1)\,{}_jX_2(z_2,z_3) \\ +{}_iX_2(z_1,z_2)\,{}_jX_1(z_3) \\ \hdashline {}_iX_2(z_1,z_2)\,{}_LU_1(z_3) \end{array}\right] \\
&+ \left[{}_qG^a_{ijk} \,\vdots\, {}_qG^b_{ijL} \,\vdots\, {}_qG^c_{iLM} \,\vdots\, {}_qG^d_{LMN}\right]\left[\begin{array}{c} {}_iX_1(z_1)\,{}_jX_1(z_2)\,{}_kX_1(z_3) \\ \hdashline {}_iX_1(z_1)\,{}_jX_1(z_2)\,{}_LU_1(z_3) \\ \hdashline {}_iX_1(z_1)\,{}_LU_1(z_2)\,{}_MU_1(z_3) \\ \hdashline {}_LU_1(z_1)\,{}_MU_1(z_2)\,{}_NU_1(z_3) \end{array}\right]
\end{aligned} \tag{7.5.10}
$$

where ${}_pX_1(z_1)$, ${}_pX_2(z_1,z_2)$, etc., are given by eqns.(7.5.5) to (7.5.7), $\Phi^a(z)=\left[zI-\phi^a(T)\right]^{-1}$ and $\phi^a(KT)=\mathcal{Z}^{-1}\{z\ \Phi^a(z)\}$, is the discrete state transition matrix. Eqns.(7.5.5) to (7.5.10) provide a state space solution to the discrete dynamic equations, if the Volterra series expansions for ${}_pX(z)$ and ${}_pY(z)$ converge. The discrete state transition equation describing the state of the system at the sampling instants may be obtained in the same way as obtained in section 7.3, for single-input single-output system. The synchronous sampled output ${}_qy(KT)$ may then be obtained from eqns.(7.5.8) to (7.5.10).

7.5.3 Synthesis of Multidimensional Z Transform Kernels

In linear system theory, it is well known that, from the state description of the continuous system, it is possible to obtain the z transfer function of the corresponding sampled-data system in terms of the discrete state transition matrix. This subsection now develops a method to determine the multidimensional z transform kernels of the multivariable system shown in Fig.7.2. A procedure for the synthesis of the multidimensional z transform kernels in terms of the discrete state transition matrix is also given.

For the system shown in Fig.7.2, the following input-output relationships exist:

$$ {}_qY_1(z) = {}_qP_{1L}(z)\,{}_LU_1(z) \tag{7.5.11}$$

$$ {}_qY_2(z_1,z_2) = {}_qP_{2LM}(z_1,z_2)\,{}_LU_1(z_1)\,{}_MU_1(z_2) \tag{7.5.12}$$

$$ {}_qY_3(z_1,z_2,z_3) = {}_qP_{3LMN}(z_1,z_2,z_3)\,{}_LU_1(z_1)\,{}_MU_1(z_2)\,{}_NU_1(z_3) \tag{7.5.13}$$

etc.

where ${}_qP_{1L}(z)$, ${}_qP_{2LM}(z_1,z_2)$, ${}_qP_{3LMN}(z_1,z_2,z_3)$ are the multidimensional z transform kernel matrices of system which are to be determined.

If the multivariable system, shown in Fig.7.2, is characterised by the dynamic equations (7.5.2) and (7.5.4), then the Volterra series expansion of ${}_qY(z)$ are given by eqns.(7.5.8) to (7.5.10). Substituting eqns.(7.5.5) to (7.5.7) in eqns.(7.5.8) to (7.5.10) yields ${}_qY_1(z)$, ${}_qY_2(z_1,z_2)$ and ${}_qY_3(z_1,z_2,z_3)$, in terms of the initial condition vector ${}_ix(0)$ and the input vector ${}_LU_1(z)$. By letting ${}_ix(0)$ as equal to zero in ${}_qY_1(z)$, ${}_qY_2(z_1,z_2)$ and ${}_qY_3(z_1,z_2,z_3)$ and then comparing the resulting equations with eqns.(7.5.11) to (7.5.13) yields the multidimensional z transform kernels, as

$$ {}_qP_{1L}(z) = {}_qE^a_I\,{}_I\phi^a_J(z)\,{}_J\phi^b_L(T) + {}_qE^b_L \tag{7.5.14}$$

$$\begin{aligned} {}_qP_{2LM}(z_1,z_2) = {} & \{{}_qE^a_I\,{}_I\phi^a_J(z_1z_2)\,{}_J\theta^a_{ij}(T)+{}_qF^a_{ij}\}\,{}_i\phi^a_K(z_1)\,{}_K\phi^b_L(T)\,{}_j\phi^a_w(z_2)\,{}_w\phi^b_M(T) \\ & +\{{}_qE^a_I\,{}_I\phi^a_J(z_1z_2)\,{}_J\theta^b_{iL}(T)+{}_qF^b_{iL}\}\,{}_i\phi^a_K(z_1)\,{}_K\phi^b_M(T) \\ & + {}_qE^a_I\,{}_I\phi^a_J(z_1z_2)\,{}_J\theta^c_{LM}(T)+{}_qF^c_{LM} \end{aligned} \tag{7.5.15}$$

$$\begin{aligned} {}_qP_{3LMN}(z_1,z_2,z_3) = {} & \{{}_qE^a_I\,{}_I\phi^a_J(z_1z_2z_3)\,{}_J\theta^a_{iK}(T)+{}_qF^a_{iK}\} \\ & \Big[{}_i\phi^a_S(z_1)\,{}_S\phi^b_L(T)\,{}_K\phi^a_w(z_2z_3)\{{}_w\theta^a_{jk}(T)\,{}_j\phi^a_T(z_2)\,{}_T\phi^b_M(T)\,{}_k\phi^a_Z(z_3)\,{}_Z\phi^b_N(T) \\ & +{}_w\theta^b_{jN}(T)\,{}_j\phi^a_T(z_2)\,{}_T\phi^b_M(T)+{}_w\theta^c_{MN}(T)\} \\ & +{}_i\phi^a_w(z_1z_2)\{{}_w\theta^a_{jk}(T)\,{}_j\phi^a_T(z_1)\,{}_T\phi^b_L(T)\,{}_k\phi^a_Z(z_2)\,{}_Z\phi^b_M(T) \\ & +{}_w\theta^b_{jM}(T)\,{}_j\phi^a_T(z_1)\,{}_T\phi^b_L(T)+{}_w\theta^c_{LM}(T)\}\,{}_K\phi^a_S(z_3)\,{}_S\phi^b_N(T)\Big] \end{aligned}$$

$$+\{{}_qE^a_I\ {}_I\Phi^a_J(z_1z_2z_3)_J\theta^b_{KN}(T)+{}_qF^b_{KN}\}\Big[{}_K\Phi^a_w(z_1z_2)$$

$$\{{}_w\theta^a_{ij}(T)_i\Phi^a_S(z_1)_S\phi^b_L(T)_j\Phi^a_T(z_2)_T\phi^b_M(T)+{}_w\theta^b_{iM}(T)_i\Phi^a_S(z_1)_S\phi^b_L(T)+{}_w\theta^c_{LM}(T)\}\Big]$$

$$+\{{}_qE^a_I\ {}_I\Phi^a_J(z_1z_2z_3)_J\psi^a_{ijk}(T)+{}_qG^a_{ijk}\}_i\Phi^a_S(z_1)_S\phi^b_L(T)_j\Phi^a_T(z_2)_T\phi^b_M(T)_k\Phi^a_Z(z_3)_Z\phi^b_N(T)$$

$$+\{{}_qE^a_I\ {}_I\Phi^a_J(z_1z_2z_3)_J\psi^b_{ijL}(T)+{}_qG^b_{ijL}\}_i\Phi^a_S(z_1)_S\phi^b_L(T)_j\Phi^a_T(z_2)_T\phi^b_M(T)$$

$$+\{{}_qE^a_I\ {}_I\Phi^a_J(z_1z_2z_3)_J\psi^c_{iLM}(T)+{}_qG^c_{iLM}\}_i\Phi^a_S(z_1)_S\phi^b_L(T)$$

$$+\{{}_qE^a_I\ {}_I\Phi^a_J(z_1z_2z_3)_J\psi^d_{LMN}(T)+{}_qG^d_{LMN}\} \qquad (7.5.16)$$

The higher order z transform kernels may be similarly obtained. It may be seen that eqn.(7.5.14) represents the z transfer function matrix of the linear system described by the first term of eqns.(7.5.2) and (7.5.4). The z transform kernels ${}_qP_{1L}(z)$ and ${}_qP_{2LM}(z_1,z_2)$ are synthesised, using the procedure described in Chapter 5, as shown in Figs.7.3(a) and (b), respectively. The third-order kernel ${}_qP_{3LMN}(z_1,z_2,z_3)$ may be similarly realised.

7.5.4 Example - Multidimensional Z Transform Kernels of the Direction Dependent System

To illustrate the method developed in section 7.5, the multi-dimensional z transform kernels of the direction dependent system of Fig. 7.1 are obtained. First, the Volterra series solution of the discrete state equation must be obtained. For the continuous part of the system, the state equation is given by

$${}_1\dot{x} = {}_1A^a_1\ {}_1x + {}_1A^b_1\ {}_1u + {}_1B^b_{11}\ {}_1x\ {}_1u + {}_1B^c_{11}\ {}_1u\ {}_1u \qquad (7.5.17)$$

where $A^a = -\omega$, $A^b = \omega$, $B^a = 0$, $B^b = -r$ and $B^c = r$. The transition matrix is given by

$$\Phi^a(s) = (sI - A^a)^{-1} = \frac{1}{(s+\omega)} \qquad (7.5.18)$$

The discrete state transition matrix, in z transform, is given by

$$z\ \Phi^a(z) = z\left[zI - \phi^a(T)\right]^{-1} = \frac{z}{(z - e^{-\omega T})} \qquad (7.5.19)$$

First order term: The first term of the Volterra series solution of the

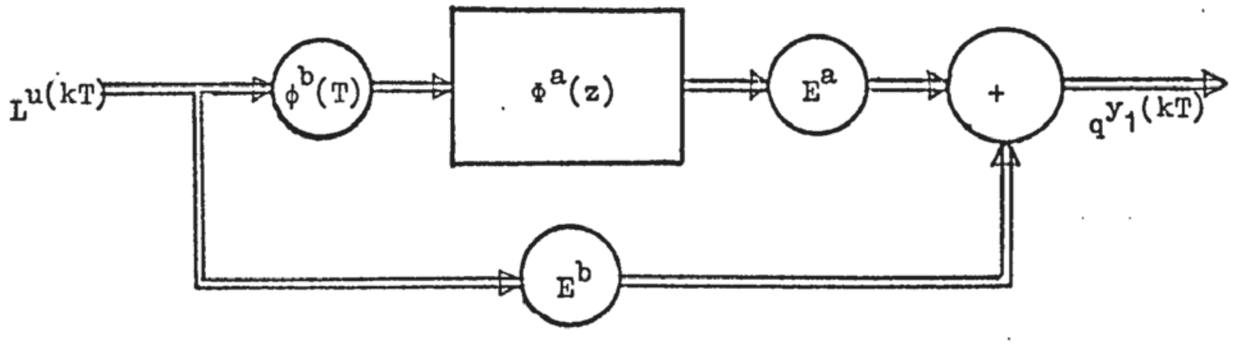


Fig.7.3(a) First-order discrete kernel, $_1P_{1L}(z)$.

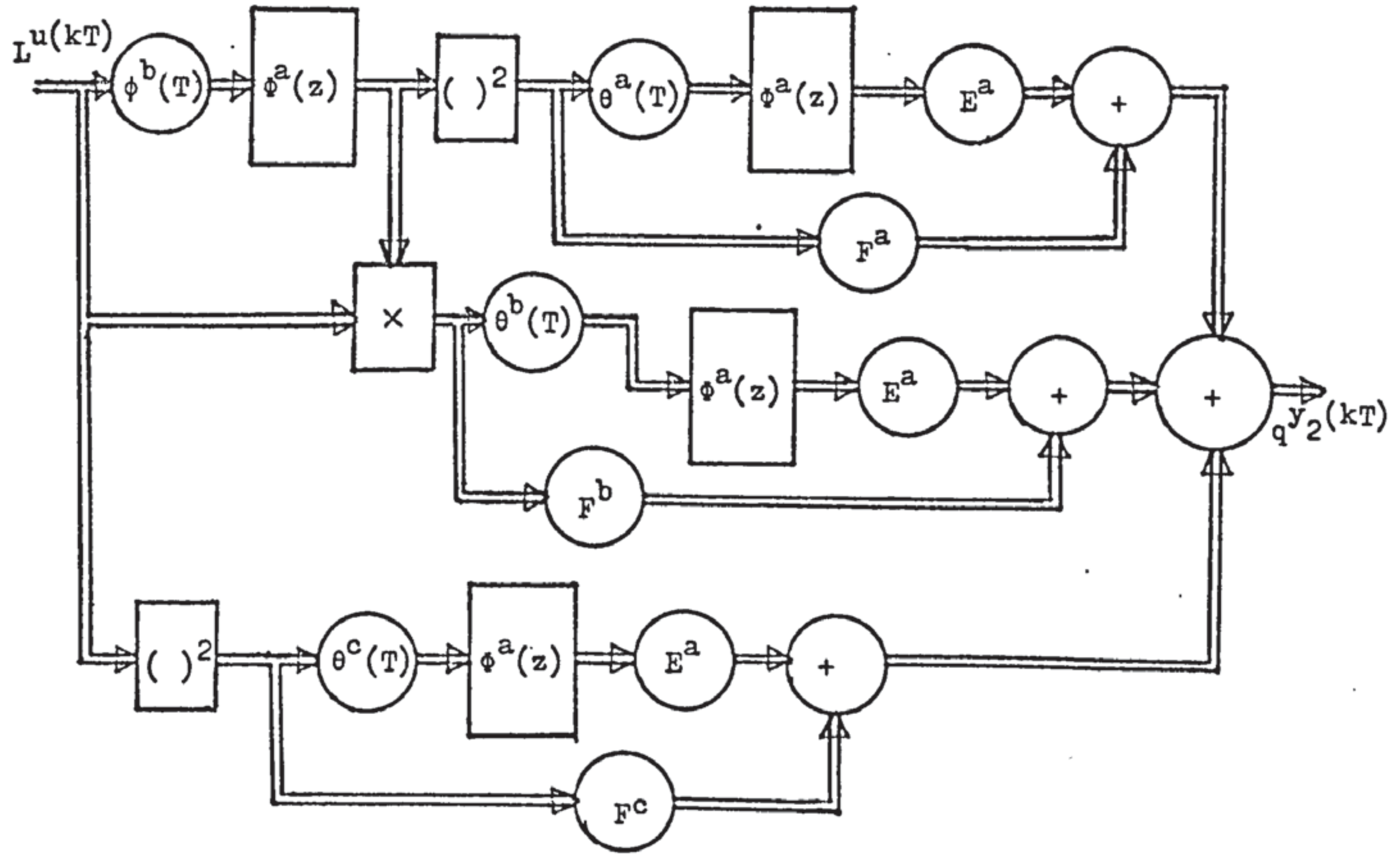


Fig.7.3(b) Second-order kernel , $_qP_{2LM}(z_1,z_2)$.

discrete state equation is then given by

$$_1X_1(z) = \{\,_1\phi_1^a(z)\; z\; _1x(0) + {}_1\phi_1^a(z)\,_1\phi_1^b(T)\,_1U_1(z)\} \tag{7.5.20}$$

where $_1\phi_1^b(T)$ is given by

$$_1\phi_1^b(T) = \int_0^T {}_1\phi_1^a(T-\tau)\;_1A_1^b\, d\tau = (1-e^{-\omega T})$$

Substituting for $_1\phi_1^b(T)$ and $_1\phi_1^a(z)$ in eqn.(7.5.20) gives

$$_1X_1(z) = \{\frac{z\;_1x(0)}{(z-e^{-\omega T})} + \frac{(1-e^{-\omega T})\,_1U_1(z)}{(z-e^{-\omega T})}\} \tag{7.5.21}$$

Second order term: To obtain the second term of the Volterra series solution, $\theta^a(T)$, $\theta^b(T)$ and $\theta^c(T)$ are to be evaluated.

$$_1\theta_{11}^a(T) = \int_0^T {}_1\phi_1^a(T-\tau)\,_1B_{11}^a\;_1\phi_1^a(\tau)\,_1\phi_1^a(\tau)\, d\tau = 0, \text{ since } {}_1B_{11}^a = 0.$$

$$_1\theta_{11}^b(T) = \int_0^T {}_1\phi_1^a(T-\tau)\,_1B_{11}^b\;_1\phi_1^a(\tau)\, d\tau = -r\, e^{-\omega T}\int_0^T d\tau = -rT\, e^{-\omega T}$$

and

$$_1\theta_{11}^c(T) = \int_0^T {}_1\phi_1^a(T-\tau)\{_1B_{11}^b\;_1\phi_1^b(\tau) + {}_1B_{11}^c\}\, d\tau = r\int_0^T e^{-\omega T}\, d\tau = rT\, e^{-\omega T}.$$

Then, the second order term is given by

$$_1X_2(z_1,z_2) = {}_1\phi_1^a(z_1z_2)\{_1\theta_{11}^b(T)\,_1U_1(z_1)\,_1X_1(z_2) + {}_1\theta_{11}^c(T)\,_1U_1(z_1)\,_1U_1(z_2)\}$$

$$= \{\frac{rT\, e^{-\omega T}(z_2-1)\,_1U_1(z_1)\,_1U_1(z_2)}{(z_2-e^{-\omega T})(z_1z_2-e^{-\omega T})} - \frac{rT\, e^{-\omega T}\, z_2\;_1x(0)\,_1U_1(z_1)}{(z_2-e^{-\omega T})(z_1z_2-e^{-\omega T})}\} \tag{7.5.22}$$

Third order term: The third order term $_1X_3(z_1,z_2,z_3)$ can be obtained after calculating $\psi^a(T)$, $\psi^b(T)$, $\psi^c(T)$ and $\psi^d(T)$.

$$_1\psi_{111}^a(T) = \int_0^T {}_1\phi_1^a(T-\tau)\left[_1B_{11}^a\{_1\phi_1^a(\tau)\,_1\theta_{11}^a(\tau) + {}_1\theta_{11}^a(\tau)\,_1\phi_1^a(\tau)\}\right] d\tau$$

$$= 0\text{ , since } {}_1B_{11}^a = 0.$$

$$_1\psi_{111}^b(T) = \int_0^T {}_1\phi_1^a(T-\tau)\,_1B_{11}^b\;_1\theta_{11}^a(\tau)\, d\tau = 0\text{ , since } {}_1\theta_{11}^a(\tau) = 0.$$

$$_1\psi_{111}^c(T) = \int_0^T {}_1\phi_1^a(T-\tau)\,_1B_{11}^b\;_1\theta_{11}^b(\tau)\, d\tau = r^2\, e^{-\omega T}\int_0^T \tau\, d\tau = \frac{r^2T^2}{2}\, e^{-\omega T}$$

$$_1\psi_{111}^d(T) = \int_0^T {}_1\phi_1^a(T-\tau)\,_1B_{11}^b\;_1\theta_{11}^c(\tau)\, d\tau = -r^2\, e^{-\omega T}\int_0^T \tau\, d\tau = \frac{-r^2T^2}{2}\, e^{-\omega T}$$

Then, the third order term is given by

$$
\begin{aligned}
{}_1X_3(z_1,z_2,z_3) &= {}_1\Phi_1^a(z_1z_2z_3)\{{}_1\theta_{11}^b(T)\,{}_1U_1(z_1)\,{}_1X_2(z_2,z_3) \\
&\quad +{}_1\psi_{111}^c(T)\,{}_1U_1(z_1)\,{}_1U_1(z_2)\,{}_1X_1(z_3)+{}_1\psi_{111}^d(T)\,{}_1U_1(z_1)\,{}_1U_1(z_2)\,{}_1U_1(z_3)\} \\
&=\{-\frac{r^2T^2\,e^{-\omega T}(z_3-1)(z_2z_3+e^{-\omega T})\,{}_1U_1(z_1)\,{}_1U_1(z_2)\,{}_1U_1(z_3)}{2(z_3-e^{-\omega T})(z_2z_3-e^{-\omega T})(z_1z_2z_3-e^{-\omega T})} \\
&\quad +\frac{r^2T^2\,e^{-\omega T}\,z_3(z_2z_3+e^{-\omega T})\,{}_1x(0)\,{}_1U_1(z_1)\,{}_1U_1(z_2)}{2(z_3-e^{-\omega T})(z_2z_3-e^{-\omega T})(z_1z_2z_3-e^{-\omega T})}\} \qquad (7.5.23)
\end{aligned}
$$

The solution of the discrete output equation can be obtained as in example 7.3.3, from eqns.(7.5.21) to (7.5.23).

The multidimensional z transform kernels of the sampled-data system can be obtained using formulas derived in section 7.5.3. Accordingly,

$$
\begin{aligned}
{}_1P_{11}(z) &= {}_1\Phi_1^a(z)\,{}_1\phi_1^b(T) \\
&= \frac{(1-e^{-\omega T})}{(z-e^{-\omega T})}
\end{aligned}
$$

$$
\begin{aligned}
{}_1P_{211}(z_1,z_2) &= {}_1\Phi_1^a(z_1z_2)\{{}_1\theta_{11}^b(T)\,{}_1\Phi_1^a(z_2)\,{}_1\phi_1^b(T)+{}_1\theta_{11}^c(T)\} \\
&= \frac{rT\,e^{-\omega T}\,(z_2-1)}{(z_2-e^{-\omega T})(z_1z_2-e^{-\omega T})}
\end{aligned}
$$

Finally, the third-order kernel is obtained as

$$
\begin{aligned}
{}_1P_{3111}(z_1,z_2,z_3) &= {}_1\Phi_1^a(z_1z_2z_3)\Big[{}_1\theta_{11}^b(T)\,{}_1\Phi_1^a(z_2z_3)\{{}_1\theta_{11}^b(T)\,{}_1\Phi_1^a(z_3)\,{}_1\phi_1^b(T) \\
&\quad +{}_1\theta_{11}^c(T)\}+{}_1\psi_{11}^c(T)\,{}_1\Phi_1^a(z_3)\,{}_1\phi_1^b(T)+{}_1\psi_{111}^d(T)\Big] \\
&= -\frac{r^2T^2\,e^{-\omega T}\,(z_3-1)(z_2z_3+e^{-\omega T})}{2(z_3-e^{-\omega T})(z_2z_3-e^{-\omega T})(z_1z_2z_3-e^{-\omega T})} \qquad (7.5.24)
\end{aligned}
$$

It may be noted that the three kernels, ${}_1P_{11}(z)$, ${}_1P_{211}(z_1,z_2)$ and ${}_1P_{3111}(z_1,z_2,z_3)$, derived here are identical with the kernels derived in Chapter 5 for direction dependent system. Now, the three kernels are synthesised as shown in Fig.7.4(a), (b) and (c), respectively.

Fig.7.4(a) Synthesis of $_1P_{11}(z)$.

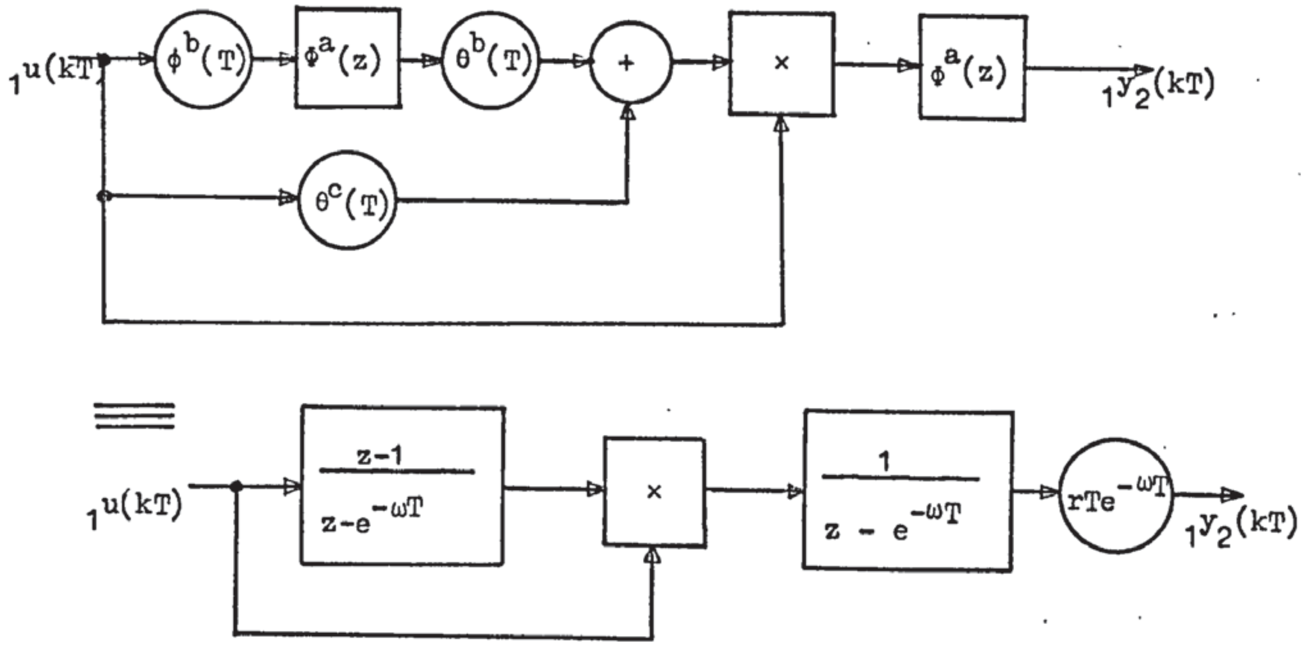


Fig.7.4(b) Synthesis of $_1P_{211}(z_1,z_2)$.

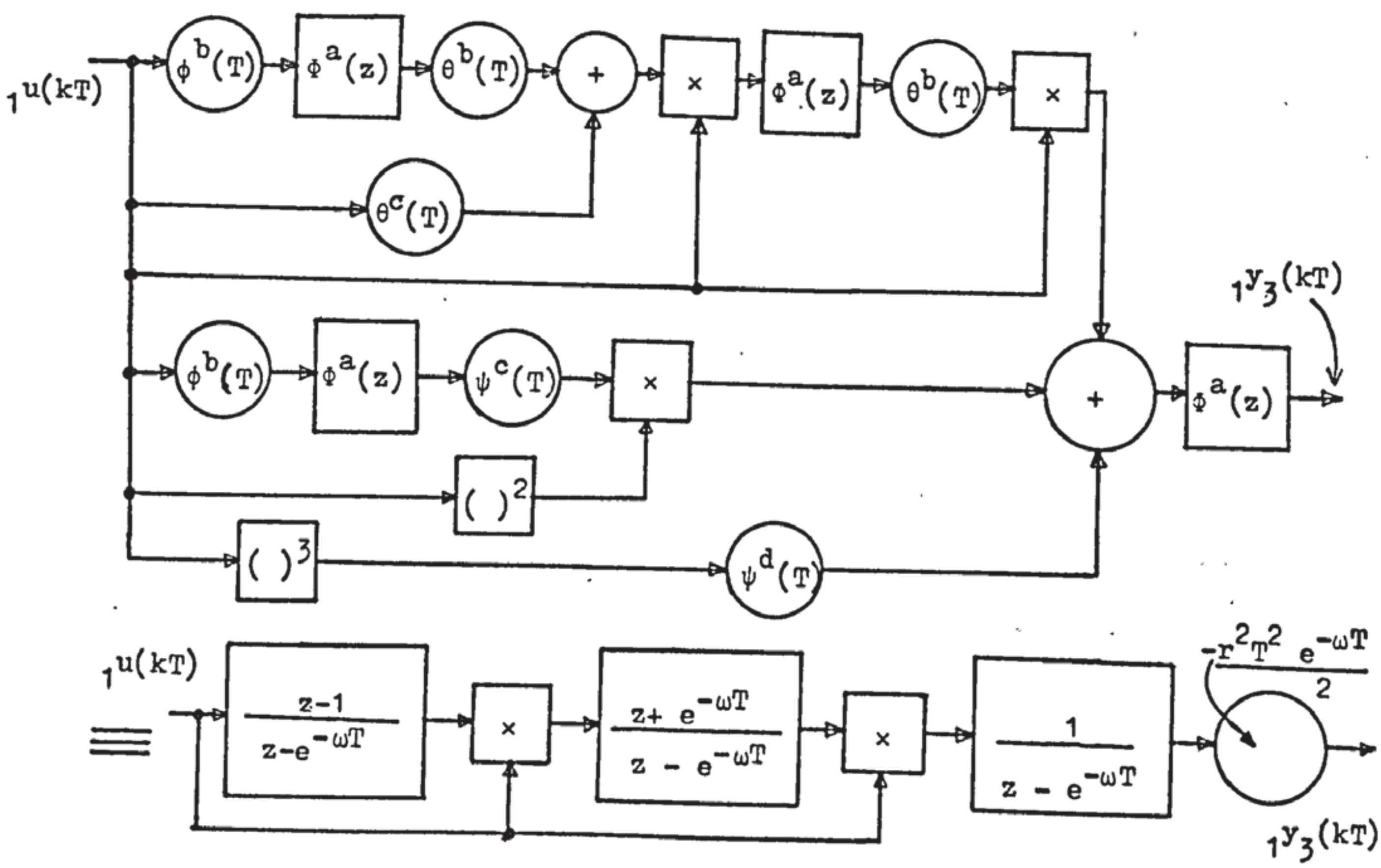


Fig.7.4(c) Synthesis of $_1P_{3111}(z_1,z_2,z_3)$.

7.6 Conclusions

It has been shown that if the input to a continuous nonlinear system is applied through a sample-and-hold device, then the discrete state equation, characterising the resulting sampled-data system, may be obtained from the state transition equation of the continuous system, by characterising the variables at the sampling instants. This system has an enormous practical significance, since it allows a continuous non-linear system to be simulated on a digital computer. The method of solution of the discrete dynamic equations may be applied to determine the sampled response of the system. However, to obtain the intersampled response of the system, the state-space solution of the asynchronous sampled-data system may be used. The validity of the method was illustrated by obtaining the discrete response of the direction dependent system of Chapter 6 and the results are identical with those obtained in Chapter 5.

It has also been shown that the solution of the discrete dynamic equations characterising the multivariable sampled-data nonlinear system yields an explicit input-output relationship in terms of the discrete state transition matrix and this relationship allows simple formulas for the multidimensional z transform kernels characterising the multivariable system, to be derived. The method of solution developed here also provides a general synthesis procedure for a multi-input, multi-output continuous nonlinear system with inputs applied through sampled-and-hold devices. The validity of the method was demonstrated by deriving and synthesising the multidimensional z transform kernels, of the direction dependent system, in terms of the linear discrete state transition matrix.

CHAPTER 8

ANALYSIS OF DISTORTION AND CROSSTALK IN FEEDBACK FM DEMODULATOR USING VOLTERRA SERIES AND FAST FOURIER TRANSFORM

8.1 Introduction

The FM demodulator with feedback was first introduced by Chaffee[96] in 1939 to reduce the distortion in FM discriminator. The feedback FM (FBFM) receiver has been used to demodulate FM signals[97] in satellite and space communications, since it has a noise threshold-extension capability, over the conventional FM discriminator, for extraction of the message from weak signals embedded in comparatively strong noise. The threshold-extension capability is achieved due to the signal-tracking property of the demodulator. The amount of the threshold-extension first increases and then decreases as the closed-loop bandwidth of the demodulator is increased[98].

In the FBFM demodulator, the frequency of the received FM signal is compressed down to some fixed intermediate frequency(IF), which allows a correspondingly narrow-bandpass filter, centered around this frequency, to be used. One of the sources of distortion in the demodulator is from this IF filter. Hoffman and Schilling[99] have obtained approximate expressions for third-harmonic distortion in the FBFM demodulator, by an iterative procedure, assuming a particular IF filter(single-tuned) and neglecting distortion due to other elements in the demodulator. Further, the iterative procedure is convergent only in the low distortion region and hence gives erroneous results in the regions where the distortion is considerably higher.

In this chapter, the nonlinear models for the measurement of nonlinear distortion, due to IF bandpass filter, the loop-limiter, the FM discriminator and the voltage controlled oscillator, are derived using Volterra series representation of the nonlinearities in the demodulator. The models, which give an explicit input-output relationship, are general and are valid for deterministic and random inputs and hence the crosstalk in multiplexed FM signal may also be determined. The distortion is measured from the spectrum of the demodulated output using the fast Fourier transform(FFT) techniques[100]. The output sequence is obtained by digital simulation of the model using the method developed in Chapter 5. The advantage of this approach is that the results obtained when the feedback gain is equated to zero correspond to those of a conventional FM discriminator. The variation of the distortion with various system parameters is studied in detail in order to obtain their optimum values for properly designing the demodulator. The distortion results obtained here are compared with those obtained by Hoffman and Schilling in order to indicate the advantages of the functional approach in achieving better results. Another advantage of the Volterra functional representation is that it enables a distortion equaliser to be obtained which, when connected to the output of the demodulator, reduces the nonlinear distortion in the demodulated output.

8.2 Nonlinear Model of the Feedback FM Demodulator

The block diagram of the feedback FM demodulator[97,99] is shown in Fig.8.1, where u(t) is the input signal given by

$$u(t) = \cos\{\omega_i t + \theta_m(t)\} \tag{8.2.1}$$

The magnitude of the input signal is normalised to unity and $\theta_m(t)$ is the phase deviation produced by the modulating signal. If $\dot{y}(t)$ is the output signal of the FBFM demodulator, then the output signal of the voltage controlled oscillator(VCO) is $\cos\{\omega_c t + q(t)\}$. The input and VCO output signals are mixed(multiplied) in a mixer, whose output signal is given by

$$r(t) = \cos\{\omega_0 t + \varepsilon(t)\} \tag{8.2.2}$$

$$\text{where} \qquad \omega_0 = (\omega_i - \omega_c) \quad \text{and} \quad \varepsilon(t) = \{\theta_m(t) - q(t)\} \tag{8.2.3}$$

It may be noted that the high frequency components such as $(2\omega_c - \omega_i)$ etc., are neglected at the mixer output. The output r(t) of the mixer is filtered by the IF bandpass filter with impulse response $j_1(\tau)$ centered at ω_0 and the filter output is given by

$$s(t) = B(t) \cos\{\omega_0 t + \delta(t)\} \tag{8.2.4}$$

where B(t) represents the amplitude fluctuations undergone by the incoming signal. The input signal to FM limiter-discriminator is s(t), whose output is $\dot{\rho}(t)$. If the amplitude B(t) of the input signal to limiter-discriminator undergoes, unusually, wide fluctuations then the loop-limiter produces an amplitude dependent phase-shift due to amplitude-to-phase modulation(AM/PM) conversion taking place in the limiter. Then, the limiter output is $\cos\{\omega_0 t + \lambda(t)\}$ where λ is given by

$$\lambda(t) = \{\delta(t) + C_p\, b(t)\} \text{ and } B(t) = e^{b(t)} \tag{8.2.5}$$

where C_p is the AM/PM conversion constant. The output $\dot{y}(t)$ of the lowpass filter $m_1(\tau)$ is then, implicitly, related to the input signal $\dot{\theta}_m(t)$.

In this section, the nonlinear model of this demodulator is derived by representing the phase nonlinearities in the demodulator by Volterra functional series. It should be noted that the phase nonlinearities due to the IF filter and the loop-limiter are only considered here and the nonlinearities due to the FM discriminator (FMD) and the Voltage Controlled Oscillator (VCO) are considered later. This is due to the fact that the nonlinear effect of the IF filter on the angle (i.e., frequency or phase) modulated signal is more severe and significant than those contributed by other nonlinearities in the demodulator.

8.2.1 Volterra Series Representation Of FBFM Demodulator

It is assumed that the VCO and the FMD are ideal at baseband and hence may be represented[101] by an integrator and a differentiator, respectively. Then, the only nonlinear effect is that contributed by the IF filter and the loop-limiter and the FBFM demodulator may be represented at baseband by the model shown in Fig.8.2, which represents a typical nonlinear feedback system, where $f(\cdot)$ is the nonlinear functional which relates the output phase $\lambda(t)$ of the limiter to the input phase $\varepsilon(t)$ of the IF filter, $m_1(\tau)$ is the impulse response of the loop lowpass filter, G is the feedback gain and

$$\dot{\rho}(t) = \dot{\lambda}(t) \qquad \text{and} \qquad \dot{q}(t) = G\,\dot{y}(t) \tag{8.2.6}$$

If $r(t) = \cos\{\omega_0 t + \varepsilon(t)\}$ is the input signal to the IF filter, then the output phase of the limiter is given by eqn.(8.2.5), where C_p is measured in degrees/db and $\delta(t)$ and $b(t)$ are given by[102]

$$\delta(t) = \mathrm{Im}\,\ln \int_0^\infty e^{j\{\varepsilon(t-\tau) - \omega_0\tau\}}\, j_1(\tau)\, d\tau = \mathrm{Im}\,\ln \int_0^\infty e^{j\varepsilon(t-\tau)}\, k_1(\tau)\, d\tau \tag{8.2.7}$$

$$b(t) = \mathrm{Re}\,\ln \int_0^\infty e^{j\varepsilon(t-\tau)}\, k_1(\tau)\, d\tau \tag{8.2.8}$$

where $k_1(\tau)$ is the impulse response of the lowpass equivalent of the IF filter $j_1(\tau)$ and is real. The system equations for the operation of the FBFM demodulator shown in Fig.8.2, may then be written in operator form as

$$\dot{y} = m_1 \otimes \dot{\rho} = m_1 \otimes \dot{\lambda}\,,\; \dot{\lambda} = f \circledast \dot{\varepsilon}\,,\; \dot{\varepsilon} = \dot{\theta}_m - \dot{q} = \dot{\theta}_m - G\,\dot{y} \tag{8.2.9}$$

where $m_1(\tau)$ is the impulse response corresponding to the transfer function $M_1(s)$ of the loop lowpass filter. Eqn.(8.2.9) gives an implicit relationship between the output $\dot{y}$ and the input $\dot{\theta}_m$ of the demodulator. An explicit input-output relationship for the demodulator may be sought in the form of Volterra series, provided that the functional $f(\cdot)$ is known. If the IF bandpass filter is assumed to have a symmetrical frequency characteristic with respect to the IF centre frequency ω_0, then the Taylor series expansion of the output phase $\delta(t)$ of the filter, as obtained from eqn.(8.2.7), will contain only odd-order terms and the IF filter may be replaced by its equivalent lowpass filter with impulse response $k_1(\tau)$. Then, the Taylor series expansion of $b(t)$, as given by eqn.(8.2.8), will contain only even-order terms. In general, the output phase $\lambda(t)$ of the limiter is given by

$$\begin{aligned} \lambda(t) &= \lambda_1(t) + \lambda_2(t) + \lambda_3(t) + \ldots\ldots \\ &= f_1 \otimes \varepsilon(t) + C_p\, f_2 \circledast \varepsilon(t) + f_3 \circledast \varepsilon(t) + \ldots\ldots \end{aligned} \tag{8.2.10}$$

where λ_1 to λ_3 may be obtained[103] from eqns.(8.2.7) and (8.2.8), as

$$\lambda_1(t) = \int_0^\infty k_1(\tau)\varepsilon(t-\tau)\, d\tau, \qquad \lambda_2(t) = -\frac{C_p}{2}\int_0^\infty k_1(\tau)\{\varepsilon(t-\tau) - \lambda_1(t)\}^2\, d\tau$$

$$\lambda_3(t) = -\frac{1}{6}\int_0^\infty k_1(\tau)\{\varepsilon(t-\tau) - \lambda_1(t)\}^3\, d\tau\,, \quad \text{etc.} \tag{8.2.11}$$

However, eqns.(8.2.10) and (8.2.11) may be conveniently represented in transform domain as

$$\Lambda = F_1(s)E(s)+ C_p\ F_2(s_1,s_2)E(s_1)E(s_2)+ F_3(s_1,s_2,s_3) \prod_{p=1}^{3} E(s_p)+\ldots \tag{8.2.12}$$

where $F_1(s)= K_1(s)$, $F_2(s_1,s_2) = -\frac{1}{2}\{K_1(s_1+s_2)- K_1(s_1)K_1(s_2)\}$ and (8.2.13)

$$F_3(s_1,s_2,s_3) = -\frac{1}{6}\{ K_1(s_1+s_2+s_3)+ 2\prod_{p=1}^{3} K_1(s_p)- 3K_1(s_1+s_2)K_1(s_3)\}$$

where $K_1(s)$ is the transfer function of the lowpass equivalent of the IF filter. Then, the Laplace transform model of the demodulator shown in Fig. 8.2, may be simplified to the approximate model shown in Fig.8.3, whose output may be characterised by the Volterra functional series. Assuming that the output is characterised by first three terms of its Volterra series expansion, we can proceed with the task of obtaining the explicit input-output kernels which characterise the system shown in Fig.8.3, by solving the system equations. If the open-loop system L shown in Fig.8.4 is the equivalent system representing the feedback model shown in Fig.8.3, then its input-output relationship is given by

$$\dot{y}(t) = L\ \{\dot{\theta}_m(t)\} \tag{8.2.14}$$

Then, solving the system equations(8.2.9) using the cascade, summing and linear system operations analogus to those described in Chapter 4, and comparing the resulting equations with the components of eqn.(8.2.14) yields the kernels of L as

$$L_1(s) = \frac{M_1(s)\ F_1(s)}{\{1+ G\ M_1(s)F_1(s)\}} \tag{8.2.15}$$

$$L_2(s_1,s_2) = \frac{C_p\ M_1(s_1+s_2)\ F_2(s_1,s_2)}{\{1+ G\ M_1(s_1+s_2)F_1(s_1+s_2)\} \prod_{p=1}^{2} \{1+ G\ M_1(s_p)F_1(s_p)\}} \tag{8.2.16}$$

$$L_3(s_1,s_2,s_3) = \frac{M_1(s_1+s_2+s_3)}{\{1+ GM_1(s_1+s_2+s_3)F_1(s_1+s_2+s_3)\}}\left[\frac{F_3(s_1,s_2,s_3)}{\prod_{p=1}^{3}\{1+GM_1(s_p)F_1(s_p)\}} - \frac{2G\ C_p\ F_2(s_1,s_2+s_3)L_2(s_2,s_3)}{\{1+GM_1(s_1)F_1(s_1)\}}\right] \tag{8.2.17}$$

where

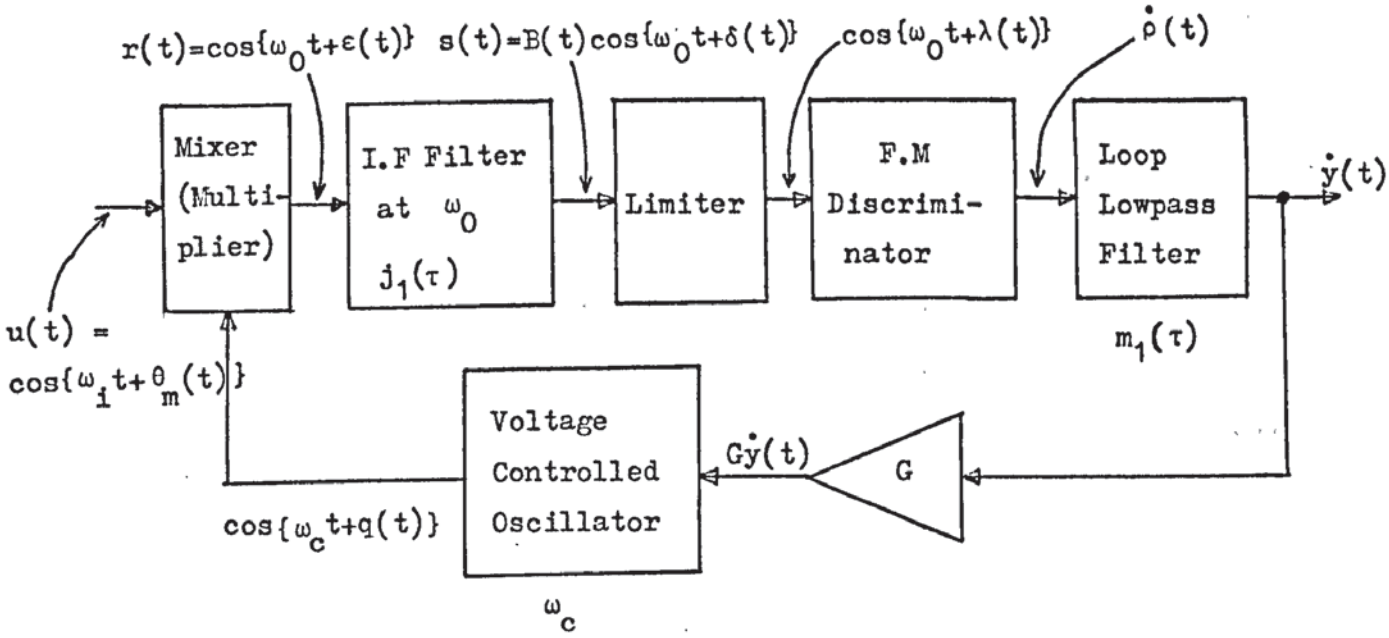


Fig.8.1 Block diagram of practical Feedback FM demodulator.

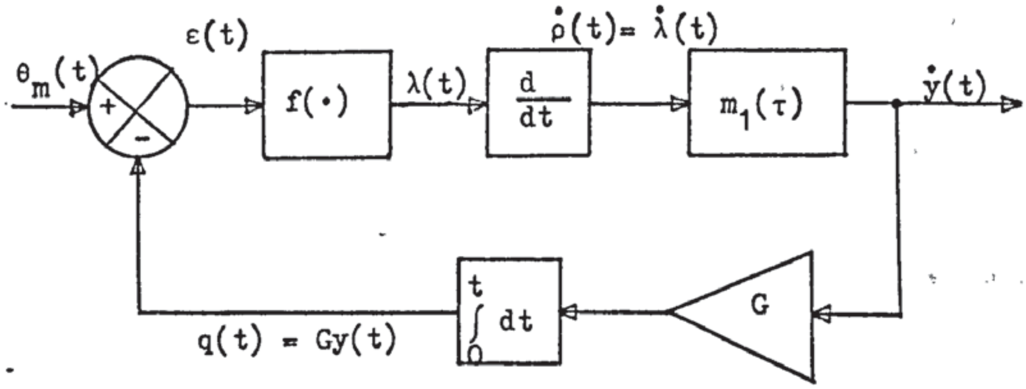


Fig.8.2 Nonlinear model of FBFM demodulator (with ideal FMD and VCO).

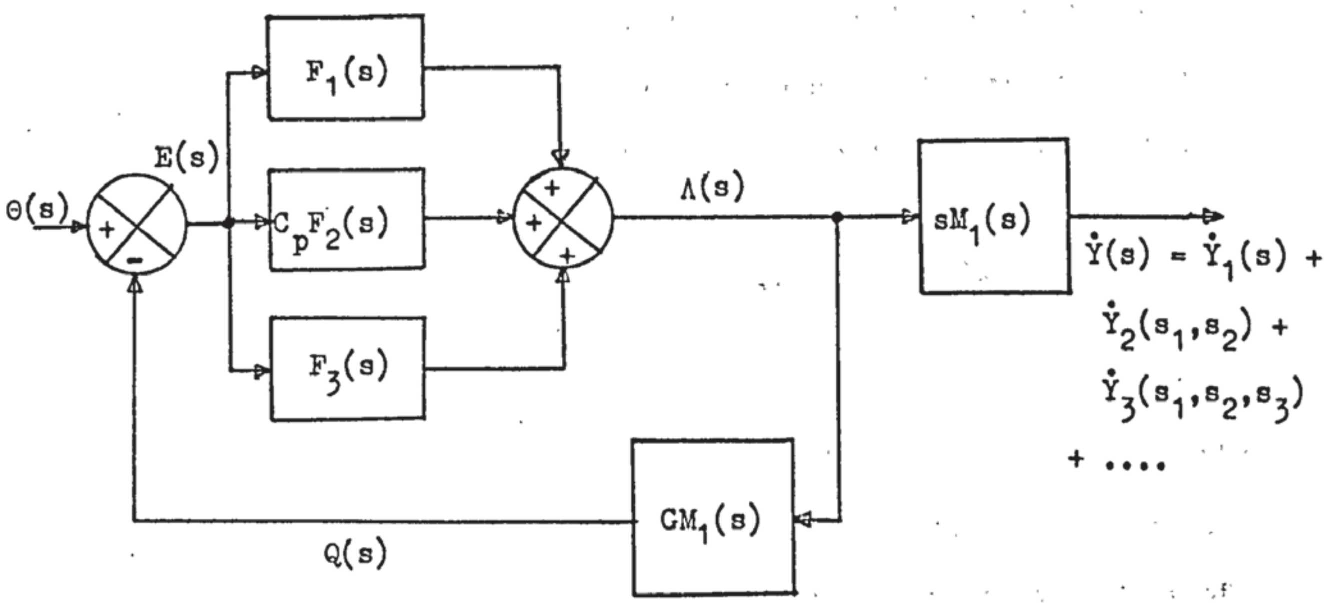


Fig.8.3 Approximate model of FBFM demodulator.

$$F_2(s_1,s_2+s_3) = -\tfrac{1}{2}\{K_1(s_1+s_2+s_3) - K_1(s_1)K_1(s_2+s_3)\} \qquad (8.2.18)$$

The linear, second and third-order continuous Volterra models of the demodulator represented by $L_1(s)$, $L_2(s_1,s_2)$ and $L_3(s_1,s_2,s_3)$, respectively, are shown in Fig.8.5.

It should be noted that the output of the linear kernel $L_1(s)$ is the desired output of the demodulator and the outputs of the second, third and higher-order kernels constitute the distortion in the demodulator output. Thus, the output of the demodulator may be written as

$$\dot{y}(t) = \dot{y}_1(t) + C_p \sum_{n=1}^{\infty} \dot{y}_{2n}(t) + \sum_{n=1}^{\infty} \dot{y}_{2n+1}(t) \qquad (8.2.19)$$

8.2.2 Simulation of the Model.

If the distortion in the demodulator output is small, then it is sufficient to consider only the first term in the summations in eqn. (8.2.19), since the contribution of the higher-order kernels to the second or third-harmonic distortion is negligibly small. Thus, the demodulator output may be approximated to $\dot{y}(t) = \dot{y}_1(t)+\dot{y}_2(t)+\dot{y}_3(t)$. The distortion may be calculated from the spectrum of the steady-state output $\dot{y}(t)$. The steady-state output is obtained by simulating the kernels $L_1(s)$, $L_2(s_1,s_2)$ and $L_3(s_1,s_2,s_3)$ and then adding their steady-state outputs.

In order to compare the results to be obtained here with those obtained by Hoffman and Schilling[99], it is necessary to assume that $C_p = 0$, so that the distortion in the demodulator is only third-harmonic, the IF bandpass filter is taken as a single-tuned filter (with a symmetric frequency characteristic) whose equivalent lowpass transfer function is given by

$$K_1(s) = \frac{\alpha}{(s+\alpha)}, \qquad (8.2.20)$$

where α is its 3 db half bandwidth, and $M_1(s)$ is taken as equal to unity (i.e., $M_1(s)$ is assumed to be absent). Further, the distortion due to limiter[97], which depends on the amplitude fluctuations at the output of the bandpass filter and on the phase distorting mechanism in the limiter circuit, is not significant compared to that contributed by the IF bandpass

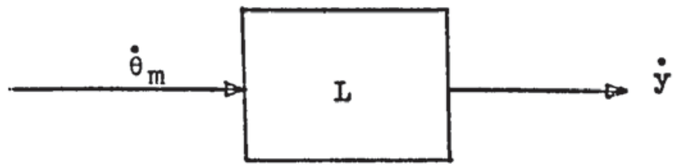


Fig.8.4 Equivalent open-loop system of Fig.8.3.

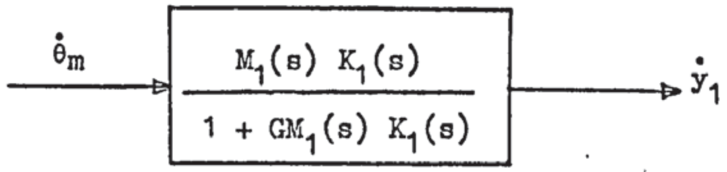


Fig.8.5(a) Linear model of FBFM demodulator.

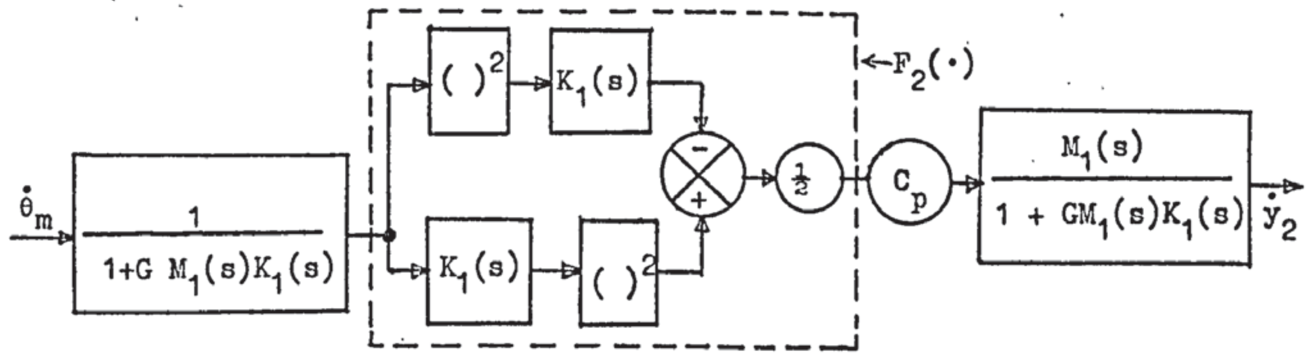


Fig.8.5(b) Second-order Volterra model.

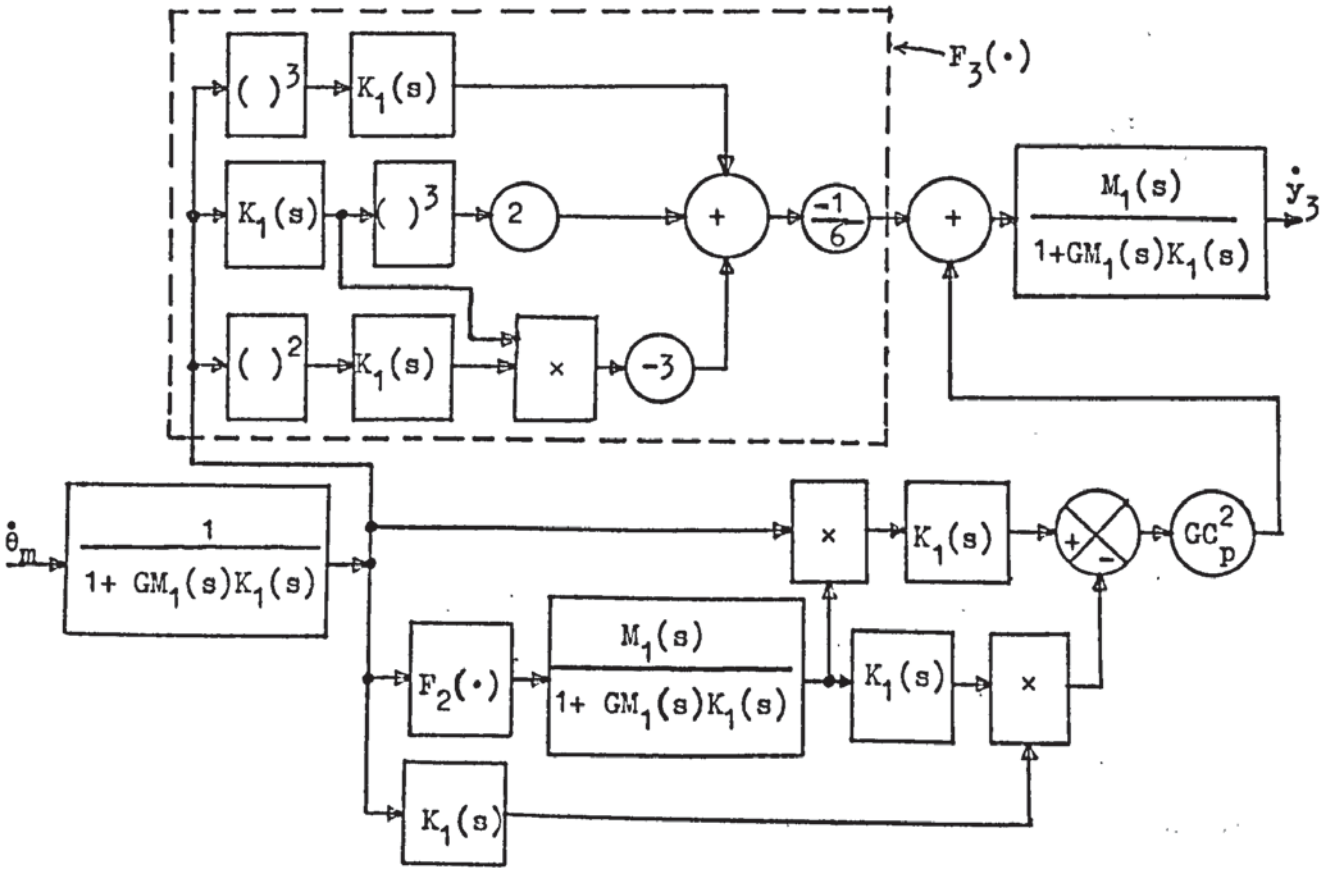


Fig.8.5(c) Third-order Volterra model.

filter. However, the models shown in Fig.8.5 are general and may be used for demodulator with a single-pole or multipole filter $M_1(s)$ following the FMD or with a multi-pole IF filter. The models may also be used to measure second-harmonic distortion due to the limiter. Then, the first and third-order kernels are given by

$$L_1(s) = \frac{\alpha}{(s+B_L)} \quad , \quad \text{and} \tag{8.2.21}$$

$$L_3(s_1,s_2,s_3) = \frac{\alpha\left[3\alpha\left(\sum_{p=1}^{3} s_p+\alpha\right)\prod_{p=1}^{2}(s_p+\alpha)-(s_1+s_2+\alpha)\left\{\prod_{p=1}^{3}(s_p+\alpha)+2\alpha^2\left(\sum_{p=1}^{3} s_p+\alpha\right)\right\}\right]}{6\left(\sum_{p=1}^{3} s_p+B_L\right)\left(\sum_{p=1}^{2} s_p+\alpha\right)\prod_{p=1}^{3}(s_p+B_L)}$$

where $B_L = \alpha(G+1)$ and the system to be simulated for distortion measurement is shown in Fig.8.6. To simulate the system shown in Fig.8.6, its discrete simulator is obtained by deriving its multidimensional z transform and then synthesising it using the procedure described in Chapter 5.

The multidimensional z transform of linear and third-order kernels associated with zero-order hold are obtained as

$$P_1(z) = \frac{\beta(1-e^{-B_LT})}{(z-e^{-B_LT})} \tag{8.2.22}$$

$$\begin{aligned}
P_3(z_1,z_2,z_3) = \frac{\beta^2}{6}\Bigg[&\frac{(z_1z_2z_3-1)}{(z_1z_2z_3-e^{-B_LT})} - \frac{(\beta+1)}{\beta(\beta-2)}\prod_{p=1}^{3}\frac{(z_p-1)}{(z_p-e^{-B_LT})} - (1-e^{-B_LT}) \\
&\left\{\frac{(z_1-1)}{(z_1-e^{-B_LT})(z_2-e^{-B_LT})} + \frac{(z_2-1)}{(z_2-e^{-B_LT})(z_3-e^{-B_LT})}\right. \\
&\left.+\frac{(z_3-1)}{(z_1-e^{-B_LT})(z_3-e^{-B_LT})}\right\} + \frac{3e^{-B_LT}(1-e^{-B_LT})}{(z_1z_2z_3-e^{-B_LT})}\prod_{p=1}^{2}\frac{(z_p-1)}{(z_p-e^{-B_LT})} \\
&-\frac{1}{\beta(\beta-2)}\prod_{p=1}^{3}\frac{(z_p-1)}{(z_p-e^{-B_LT})}\left\{\frac{(\beta^2-3\beta-1)(z_1z_2z_3-e^{-3B_LT})}{(z_1z_2z_3-e^{-B_LT})}\right. \\
&\left.+\frac{e^{-B_LT}(1-e^{-\alpha T})(z_1z_2-e^{-2B_LT})}{(z_1z_2-e^{-\alpha T})(z_1z_2z_3-e^{-B_LT})}\right\}\Bigg]
\end{aligned} \tag{8.2.23}$$

where $\beta = \frac{\alpha}{B_L}$. The discrete simulator, for the system shown in Fig.8.6, represented by $P_1(z)$ and $P_3(z_1,z_2,z_3)$ is synthesised, using the procedure described in Chapter 5, as shown in Fig.8.7, where

$$C_1 = \beta(1-e^{-B_LT}) \;,\quad C_2 = \frac{-(\beta+1)}{\beta(\beta-2)} \;,\quad C_3 = -(1-e^{-B_LT}),\quad C_4 = 3e^{-B_LT}(1-e^{-B_LT}),$$

$$C_5 = \frac{(\beta^2-3\beta-1)(1-e^{-3B_LT})}{\beta(\beta-2)} \;,\quad C_6 = \frac{e^{-B_LT}(1-e^{-\alpha T})}{\beta(\beta-2)} \;,\quad C_7 = \frac{(\beta^3-3\beta-1)}{\beta(\beta-2)} \tag{8.2.24}$$

It may be noted that the simulator represents the input-output relationship of the demodulator in terms of the design parameters α and B_L of the FBFM demodulator.

8.3 Distortion in FBFM Demodulator

The distortion in FBFM demodulator is of two types: (a) Harmonic distortion and (b) Intermodulation distortion(or crosstalk). The harmonic distortion occurs in a single communication channel and is due to nonlinearities in the demodulator which act on the received signal to produce some disturbing signals in addition to the desired(linear) signal. The intermodulation distortion or crosstalk occurs between two different communication channels as, for example, when the received signal is demodulated, the nonlinearities in the demodulator cause the signals in different channels to beat against one another resulting in the generation of new, unwanted signals which appear at the output of a desired channel. In this section, the results of the third harmonic distortion and the crosstalk, obtained from the simulation of the model, are presented.

8.3.1 Harmonic Distortion for Sinusoidal Input Signal

When the modulating signal $\dot{\theta}_m$ is a single sine wave given by

$$\dot{\theta}_m(t) = \gamma \sin \omega_m t \tag{8.3.1}$$

where γ is the magnitude and f_m is the frequency of the modulating signal, the distortion may be computed from the spectrum of the output of the demodulator. The input signal is sampled at a rate much higher than the Nyquist rate and fed to the simulator shown in Fig.8.7. The steady-state output sequence $y(iT)$ of the simulator is then transformed using the

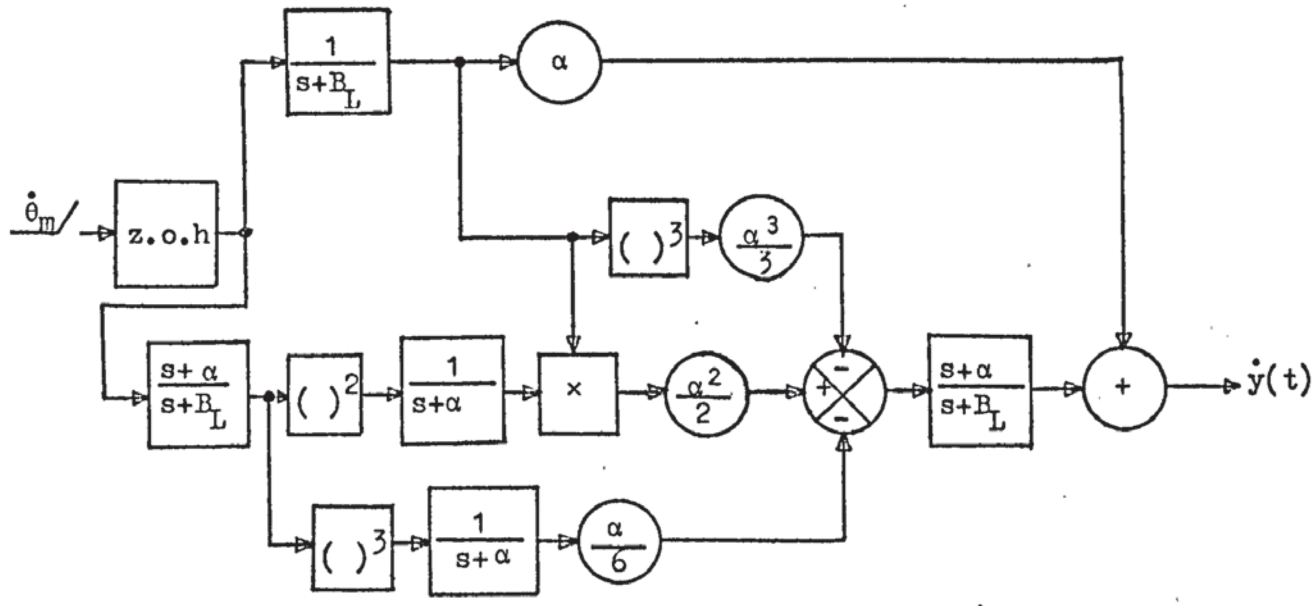


Fig.8.6 System to be simulated for distortion measurement.

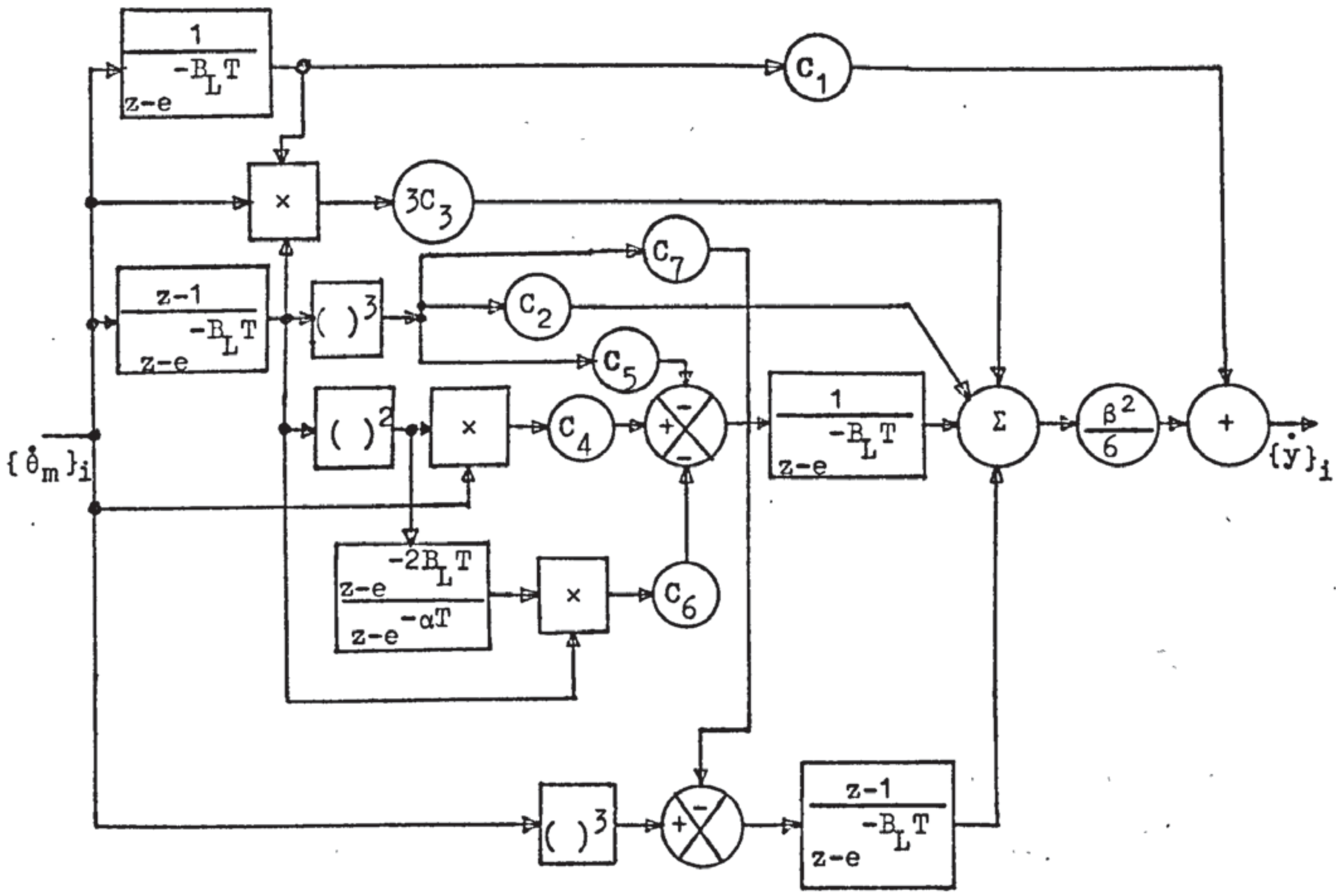


Fig.8.7 Discrete simulator for FBFM demodulator.

discrete Fourier transform[100], which gives the spectrum of the output, at discrete points $\omega = \frac{2\pi p}{NT}$, as

$$Y(\frac{2\pi p}{NT}) = \frac{1}{N} \sum_{i=0}^{N-1} y(iT) \exp(-j \frac{2\pi ip}{N}) \quad , \quad p = 0,1,2,\ldots,(N-1) \qquad (8.3.2)$$

where $\omega_s = \frac{2\pi}{T} = N\omega_m$, is the sampling frequency, $\omega_m = \frac{2\pi}{NT}$, is the input signal frequency and T is the sampling interval. Thus, $|Y(\frac{2\pi p}{NT})|$, for p=1, represents the magnitude of the desired fundamental frequency component $Y(\omega_m)$ and $|Y(\frac{2\pi p}{NT})|$, for p=3, represents the magnitude of the third-harmonic component $Y(3\omega_m)$, in the output. But, one point remains to be made before proceeding to calculate the distortion. The input sequence to the system is applied through a zero-order hold and hence the true spectrum of the output will be equal to the output spectrum divided by the frequency response of the zero-order hold. The frequency response of the zero-order hold is given by

$$H_1(\frac{2\pi p}{NT}) = T\, e^{-j\pi p/N} \frac{\mathrm{Sin}(\pi p/N)}{(\pi p/N)} , \quad p = 0,1,2,\ldots\ldots,(N-1) \qquad (8.3.3)$$

and the true output spectrum is given by

$$Y'(\frac{2\pi p}{NT}) = \frac{Y(2\pi p/NT)\ (\pi p/NT)\ e^{j\pi p/N}}{\sin(\pi p/N)} , \quad p = 0,1,2,\ldots,(N-1) \qquad (8.3.4)$$

The third-harmonic distortion D_3 is defined as the ratio of the magnitude of the third-harmonic component to the magnitude of the fundamental frequency component, and is given by

$$D_3 = \frac{|Y'(6\pi/NT)|}{|Y'(2\pi/NT)|} = \frac{|Y(6\pi/NT)|\, 3\sin(\pi/N)}{|Y(2\pi/NT)|\sin(3\pi/N)} \qquad (8.3.5)$$

where N is the number of samples in the input signal. In the distortion measurement carried out here, N is chosen to be large (i.e., N=200) in order to keep the aliasing error due to sampling, to a minimum. Then, D_3 may be approximated to

$$D_3 \simeq \frac{|Y(6\pi/NT)|}{|Y(2\pi/NT)|} \qquad (8.3.6)$$

Singleton's mixed-radix FFT has been used for computing the spectrum of the output. The distortion measurement was carried out for various values of the system parameters α and G and input signal parameters γ and ω_m. The results are presented here in three sets so that the variation of the distortion with the system parameters can be studied precisely.

The first set gives the plot of the distortion D_3 versus feedback gain G, for various values of the amplitude γ of the input signal, and is shown in Fig.8.8, where $A = (\alpha/\omega_m)$, is the ratio of the 3 db half-bandwidth of the IF filter to the input signal frequency. The graphs clearly show that the distortion decreases as G increases. It is interesting to note that the results for G=0 correspond to the case of ordinary FM discriminator preceded by an RF filter[104] and is considered later. These graphs also show, in general, that the distortion increases with γ and decreases with A. In particular, it may be noted that when γ is large the distortion, for $A < 1$, first increases with G and then decreases. This may be due to the fact that, in the range $0 < G < 10$, the effect of the third-order nonlinearities is predominant. Another reason for this may be that, when the loop is closed(i.e., $G > 0$), the filter bandwidth α is less than ω_m. Thus, in the range $0 < G < 10$, α must be greater than ω_m(i.e.,$A > 1$) in order to obtain low distortion.

Fig.8.9, which shows the second set of results, contains the plot of D_3 against A for various γ's. The curves indicate that the 3 db half-bandwidth of the IF filter $\alpha/2\pi$ must be greater than the input modulation frequency f_m in order to achieve low distortion. Again, the graphs for G=0, which correspond to open-loop FM discriminator, show that, for $\gamma > 2$, the distortion first increases with A and then decreases. This points out that the input signal amplitude must be small for this case in order to obtain low distortion.

The third set of results are shown in Fig.8.10, which shows the variation of distortion with γ, for various A's. The graphs, in general, show that the distortion increases with γ and therefore the amplitude of the modulating signal must be small in order to achieve low distortion.

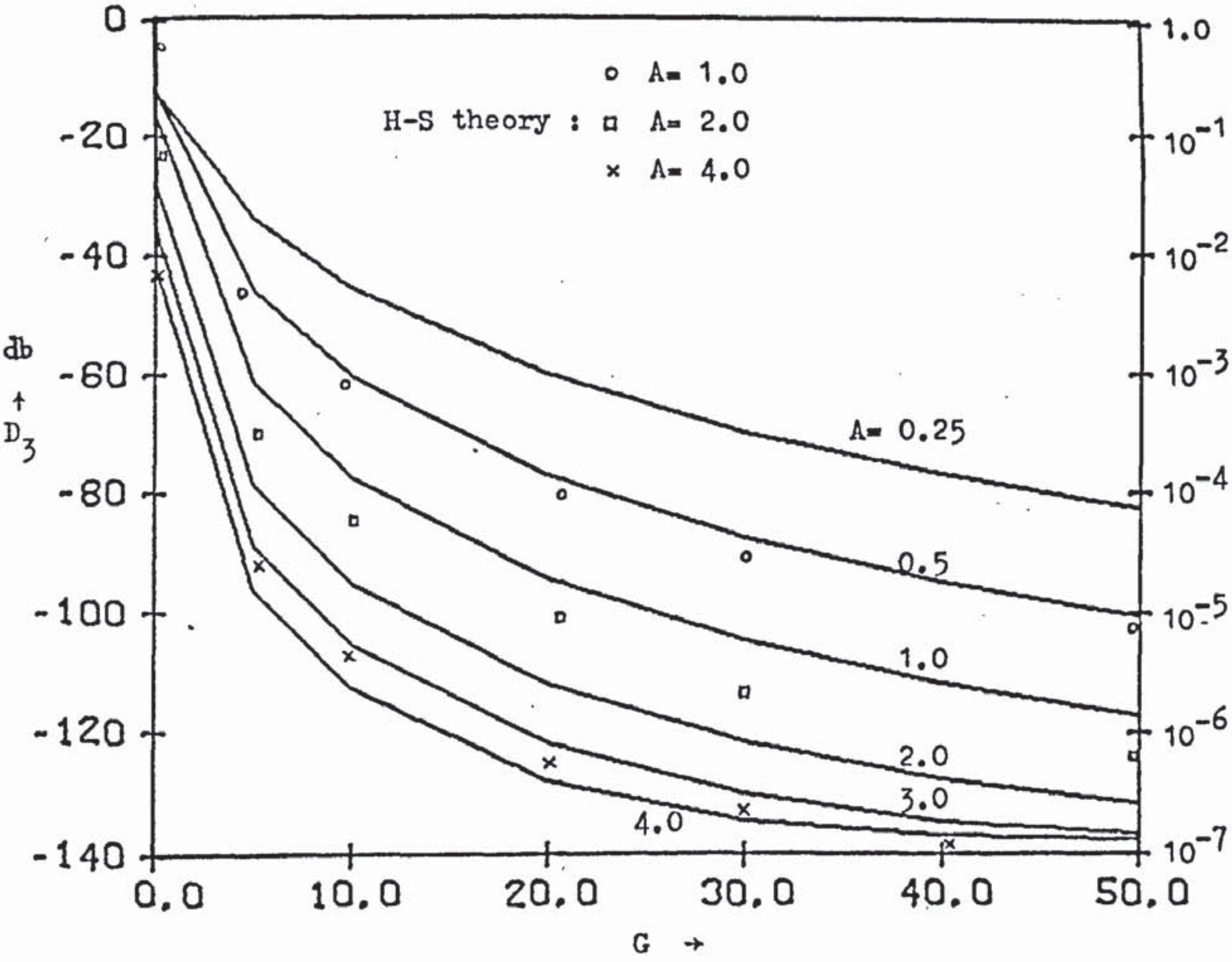


Fig.8.8(a) Distortion versus feedback gain, $\gamma = 2$.

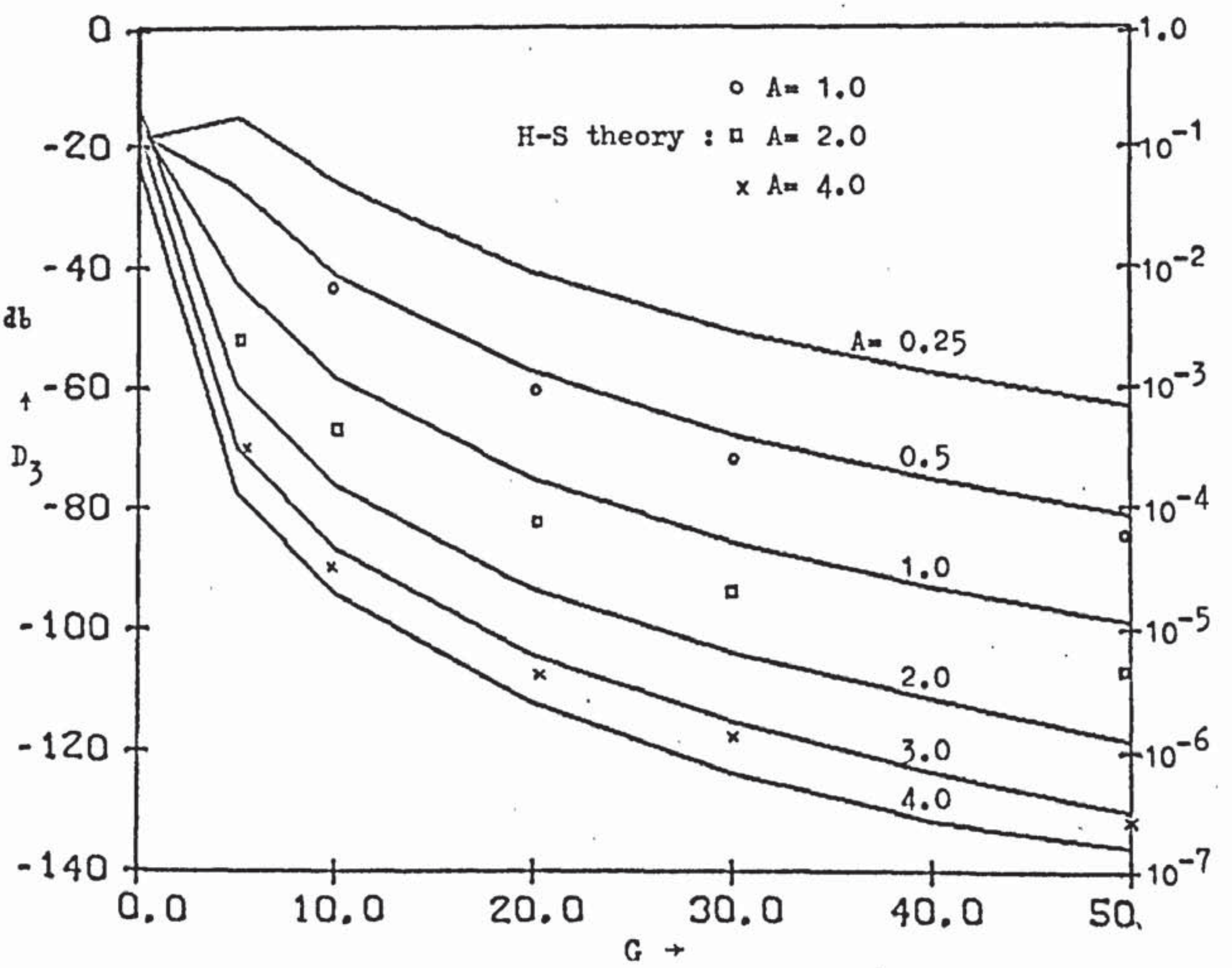


Fig.8.8(b) Distortion versus feedback gain G, $\gamma = 6$.

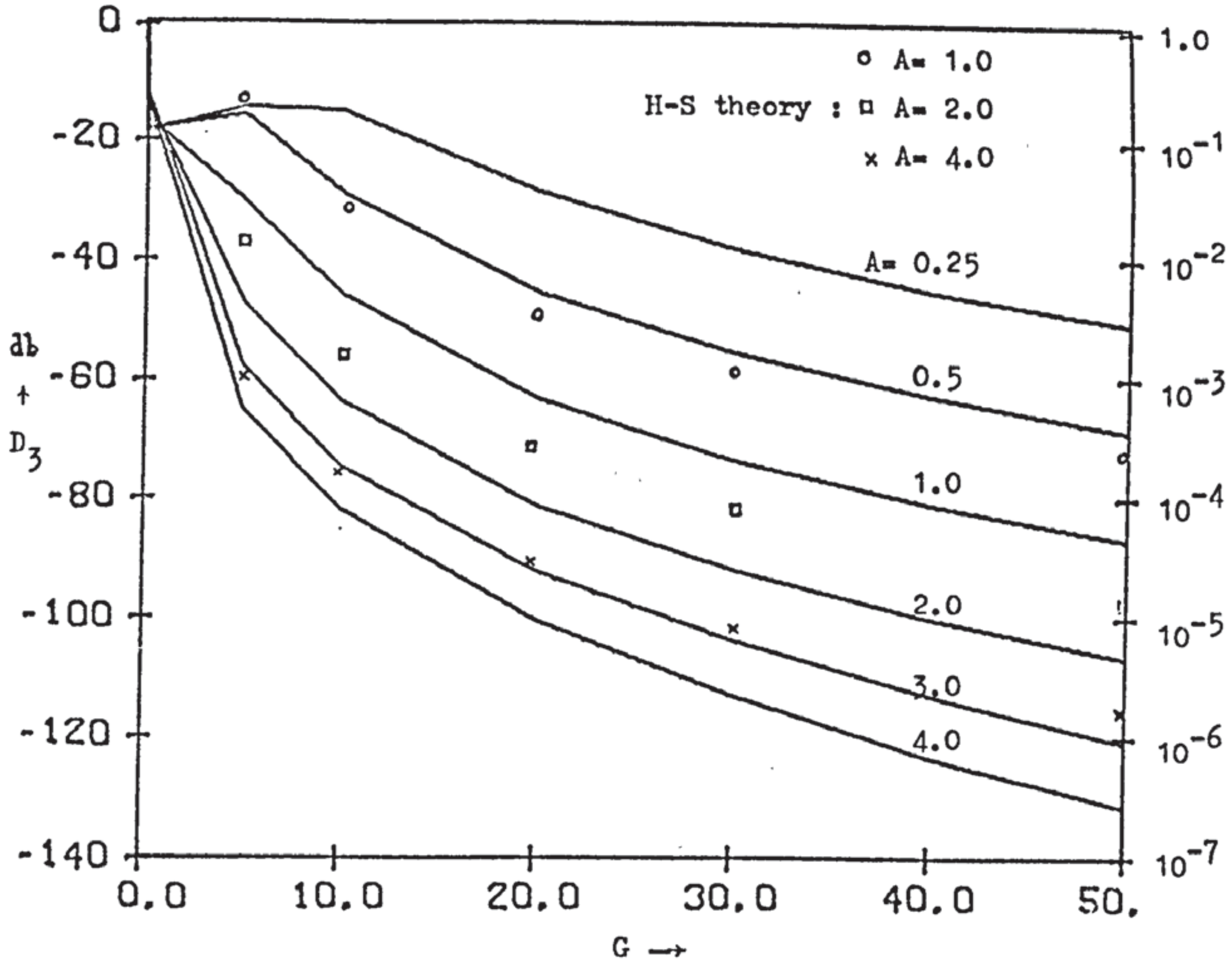


Fig.8.8(c) Distortion versus feedback gain G, γ = 12 .

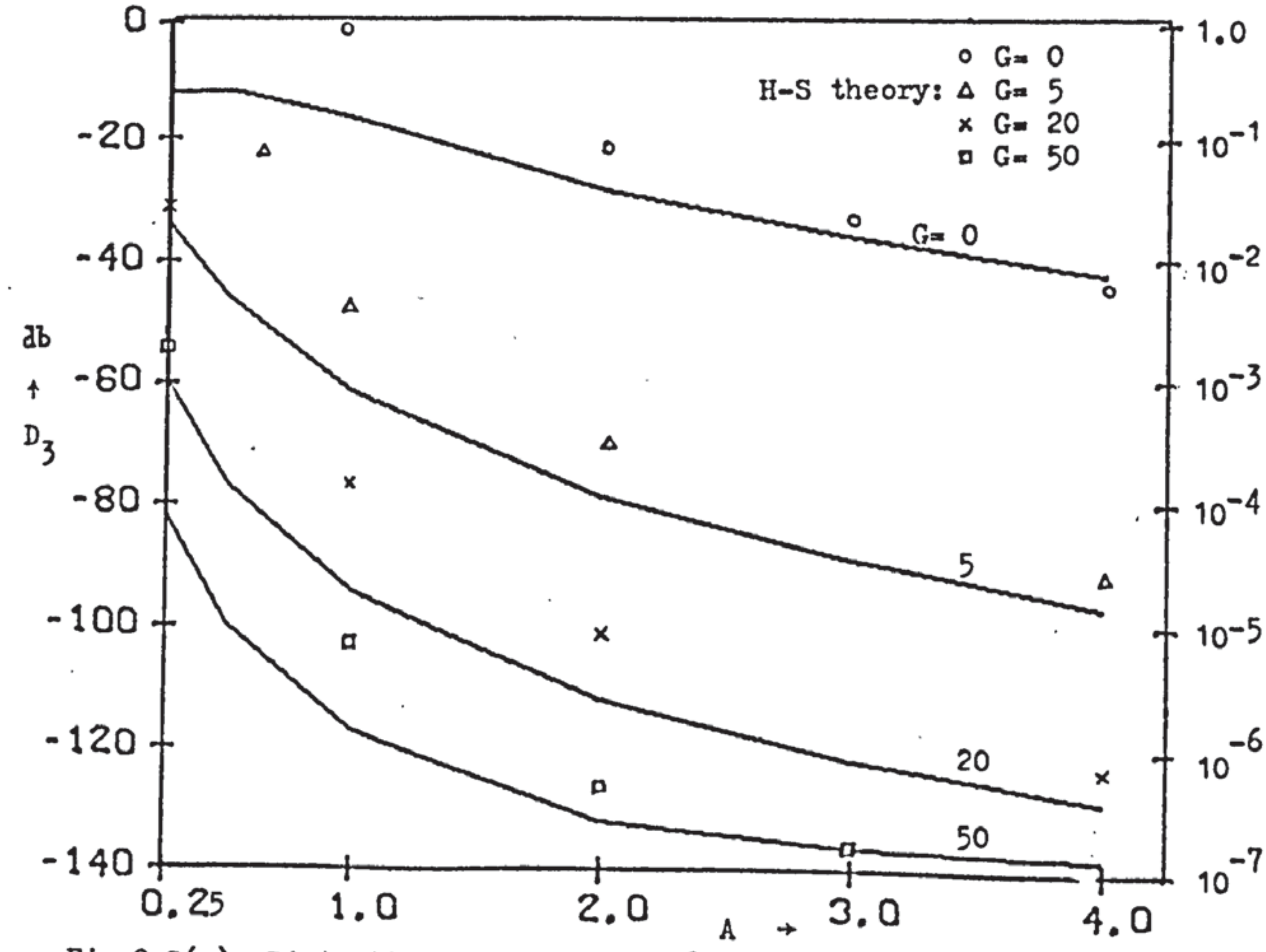


Fig.8.9(a) Distortion Versus A, γ = 2 .

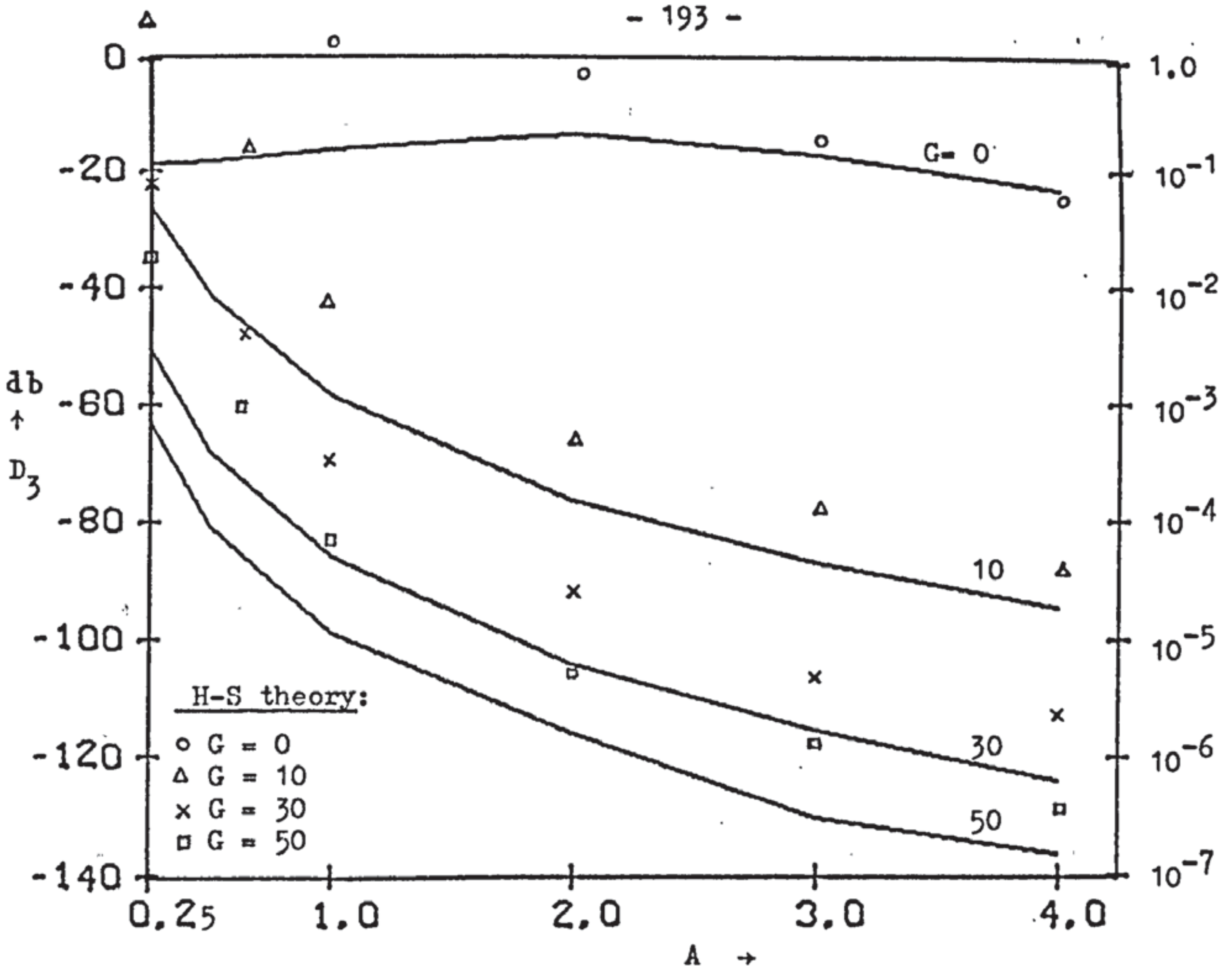


Fig.8.9(b) Distortion versus A, $\gamma = 6$.

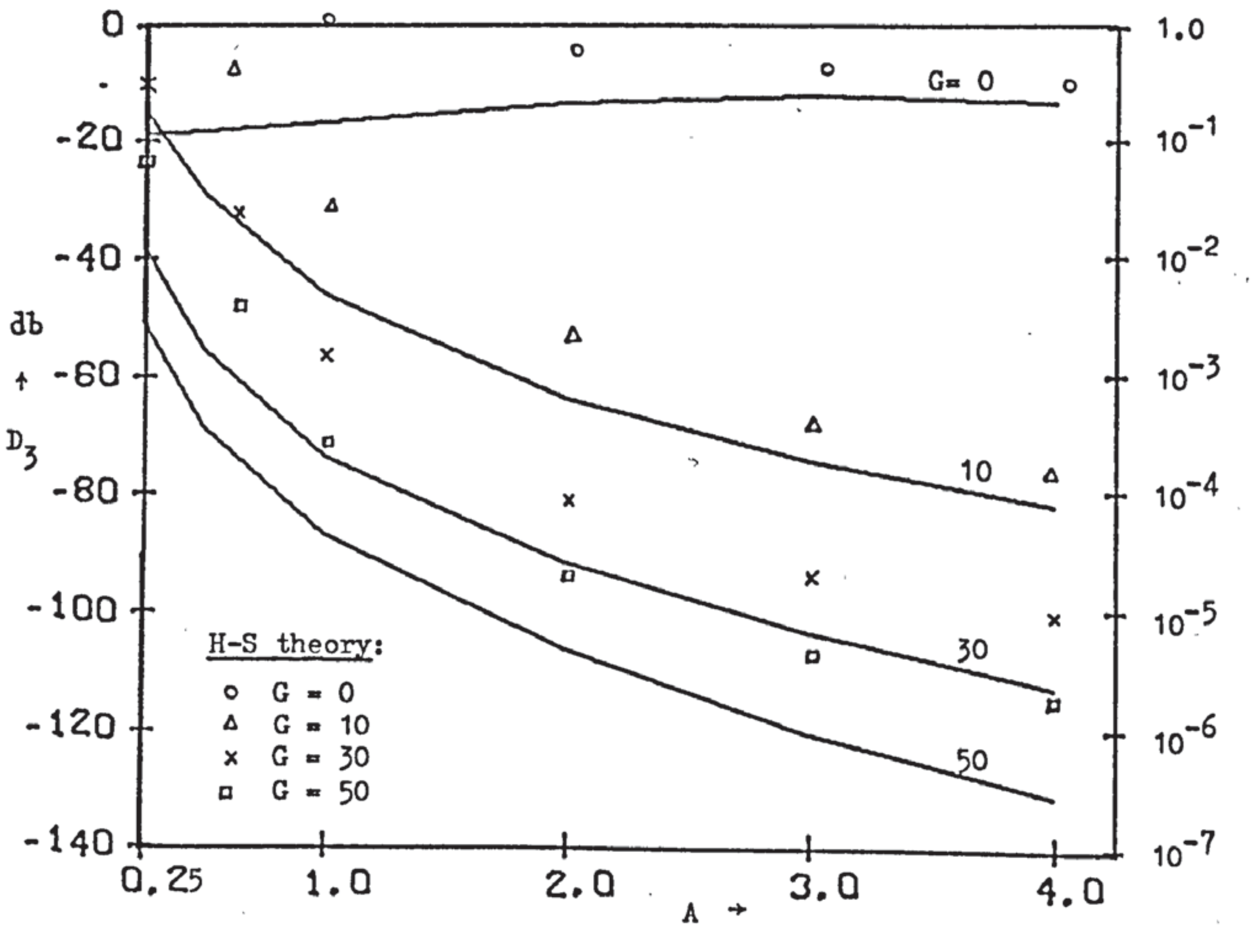


Fig.8.9(c) Distortion versus A, $\gamma = 12$.

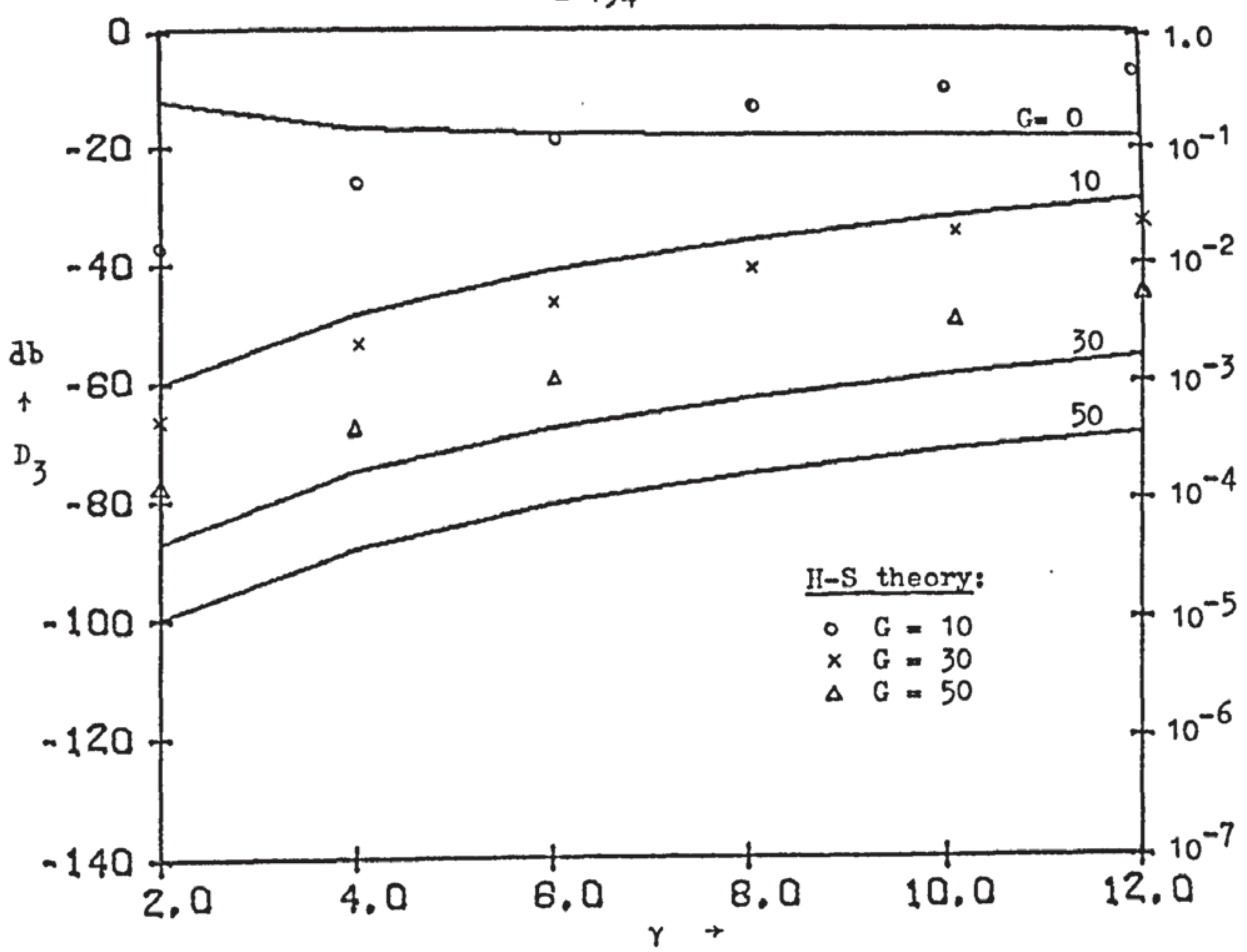


Fig.8.10(a) Distortion versus input amplitude, A = 0.5 .

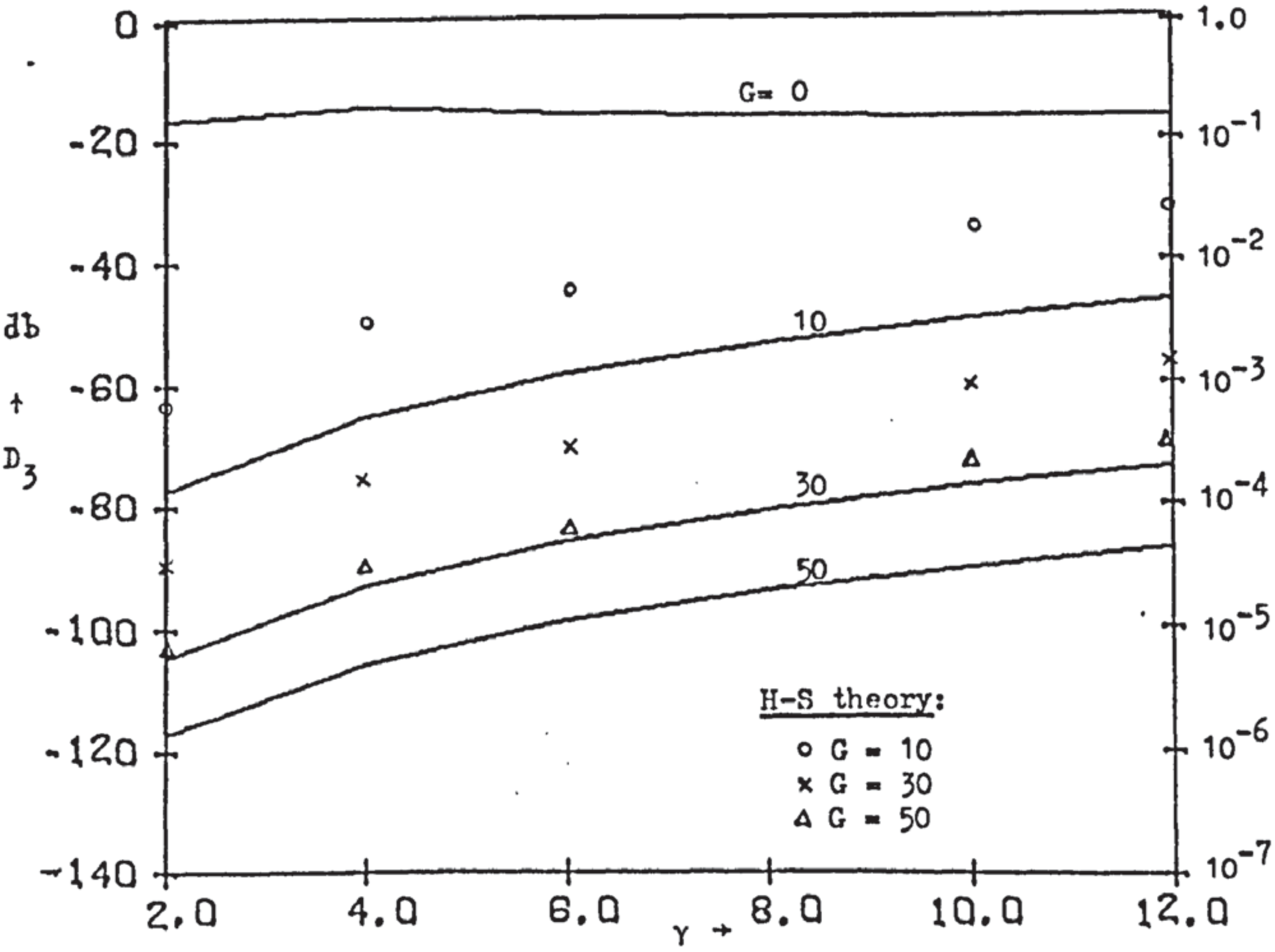


Fig.8.10(b) Distortion versus input amplitude, A = 1.0 .

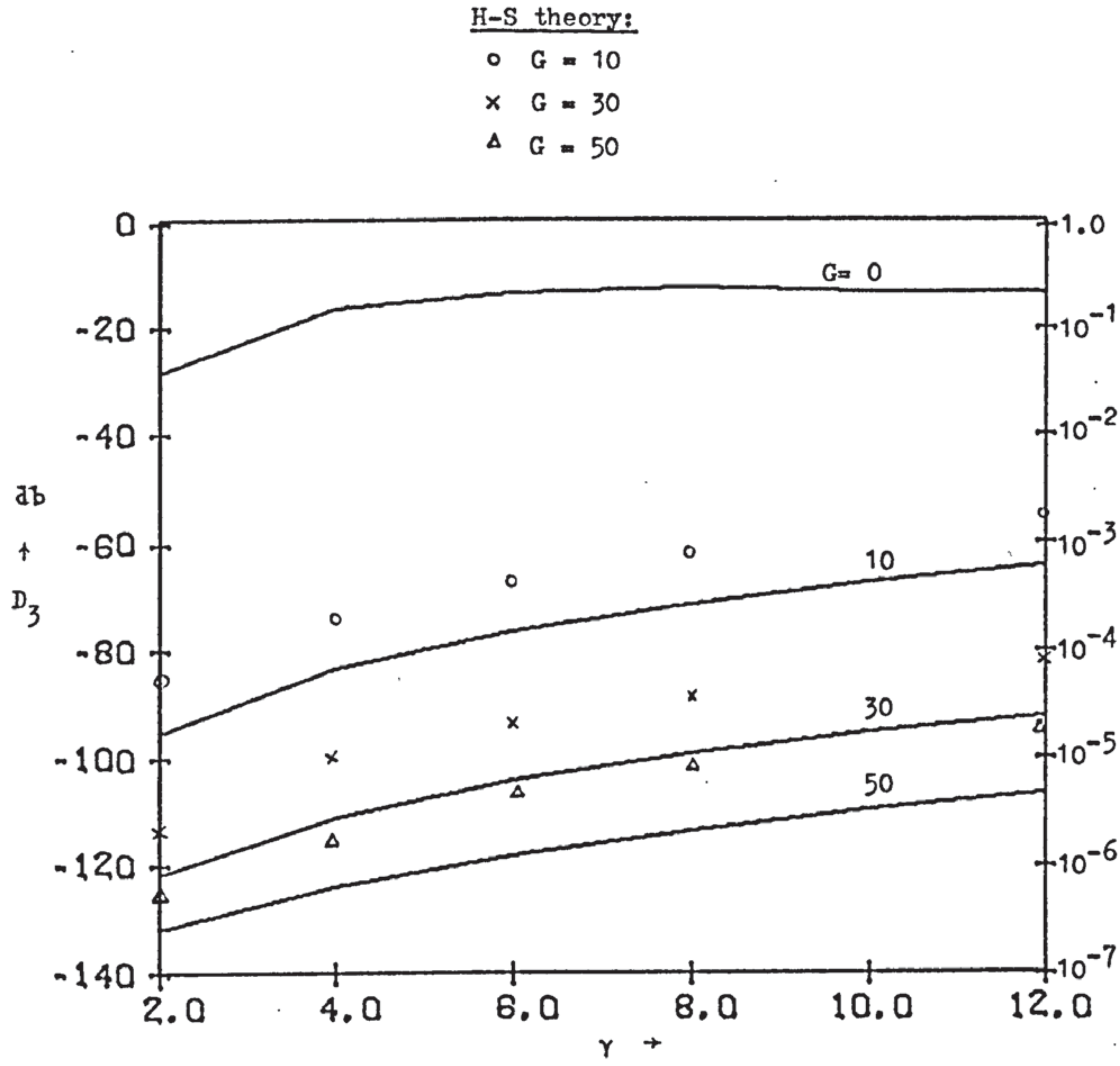


Fig.8.10(c) Distortion versus input amplitude, A = 2.0 .

Fig.8.11 shows the variation of k with γ, where k, which is the ratio of the peak frequency deviation produced by the modulating signal to the small signal bandwidth of the linearised FBFM demodulator, is given by

$$k = \frac{\gamma\omega_m}{\alpha(G+1)} = \frac{\gamma}{A(G+1)} \tag{8.3.7}$$

The results show that, for distortion in the range $-60 < D_3 < -140$ db and for $\gamma < 100$, the value of k is less than unity, which indicate that the FBFM demodulator, in order to ensure low distortion, has a wider bandwidth than the peak frequency deviation produced by the modulation.

It was mentioned earlier that if the feedback gain G is made to approach zero, the FBFM demodulator degenerates to an ordinary FM discriminator preceded by an RF filter of half bandwidth α and $B_L \to \alpha$. Fig.8.12 shows the variation of the third-harmonic distortion of the FMD with A. The results show that α must be greater than $5\omega_m$ in order to obtain an acceptable distortion level and that the input signal amplitude must be small to ensure low distortion. This clearly brings out the disadvantages of the ordinary discriminator for demodulation of FM signals, because if $\alpha \geq 5\omega_m$, the noise power that will appear at the output of IF filter may be much higher than the signal power and hence the message extracted from the demodulator output does not meet the requirements of high fidelity transmission.

Now, the results obtained here are compared with those obtained by Hoffman-Schilling, in order to establish that the present method, using Volterra series, yields lower values of distortion. Comparing the results of D_3 versus G, shown in Fig.8.8, it is clear that for $A < 4$, Hoffman-Schilling theory gives much higher distortion than that predicted by the present theory. The comparison of the results of D_3 versus A, shown in Fig.8.9, reveals that the results of both the theories are almost the same only for $\gamma \leq 2$ and $A > 4$ and hence their theory is inadequate for large input signals. Fig.8.10 shows that their theory yields low distortion only for the case when $A > 4$, $G > 50$, $\gamma \leq 2$ and does not give correct results in the region where

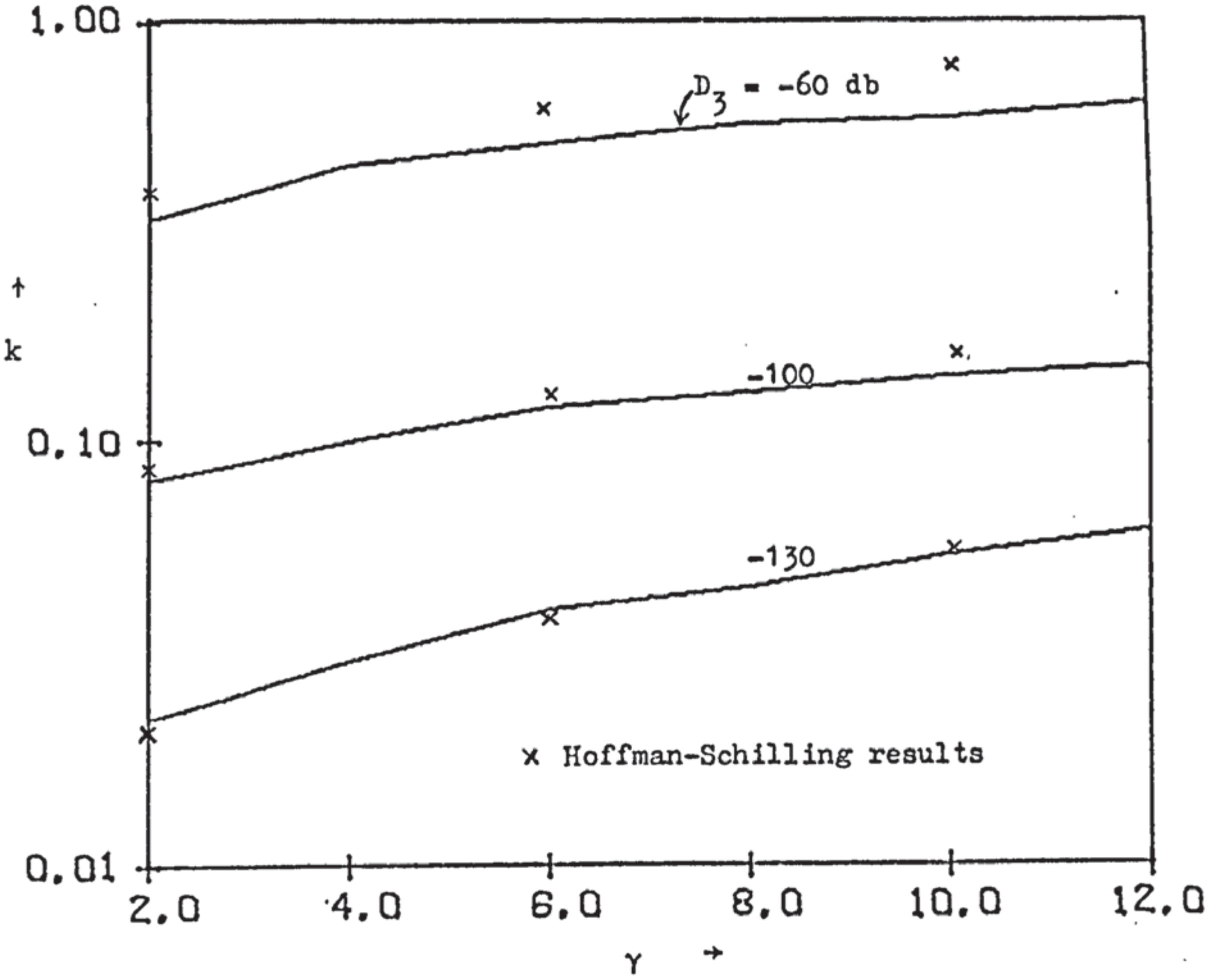


Fig.8.11 Normalised frequency deviation versus input amplitude.

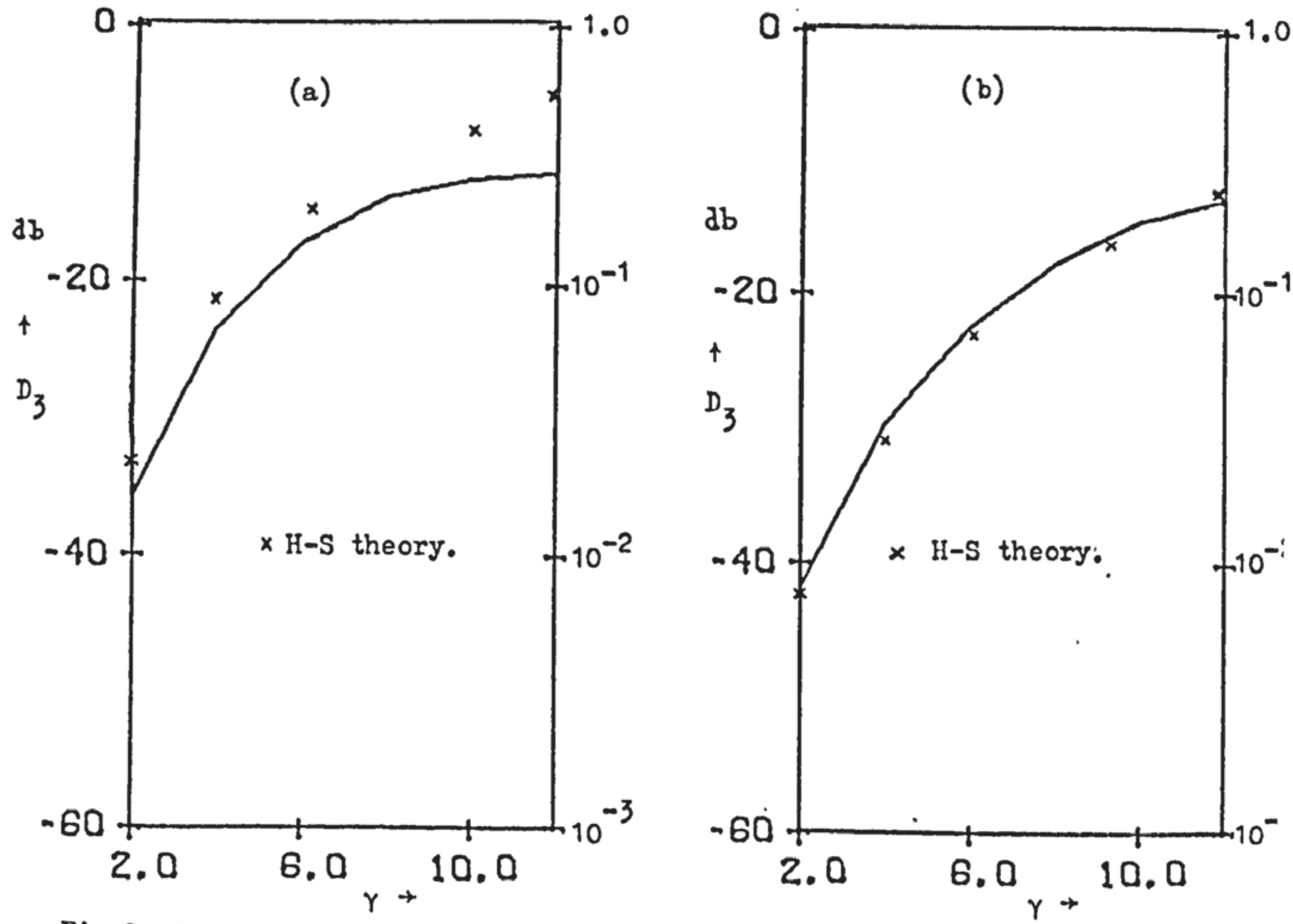


Fig.8.12 Distortion in FM discriminator versus γ, G=0. (a) A=3.0 (b) A=4.0.

the distortion is considerably higher. The discrepancy in the results is due, no doubt, to various assumptions and approximations made in their theory. However, their theory does predict better results than the present theory in the region where the distortion is low. This is evident from the results shown in Fig.8.11 in the range below -140 db. This is because the assumption that $|\dot{B}/\alpha| << B$, made in their theory, is valid only in this range. Finally, the comparison of the results, shown in Fig.8.12 for the degenerate FBFM case, indicates that for $A > 3.0$ and $\gamma \leq 6$, the results of both theories are almost the same.

The following optimum system parameters may then be suggested for achieving low distortion and for properly designing the FBFM demodulator: $\gamma \leq 2.0$, $A \geq 3.0$ and $G \geq 40$.

Now, the results may be extended to the case when the filter $M_1(s) = \frac{\omega(s+\alpha)}{\alpha(s+\omega)}$, is present and also for the case when IF filters are multi-pole filters.

8.3.2 <u>Crosstalk in FDM/FM signal</u>

Crosstalk is also known as "Inter-modulation distortion" and occurs when the input signal consists of two or more channels in frequency-division multiplex. For instance, if $\dot{\theta}_{m1}(t) = \gamma_1 \sin \omega_{m1} t$ is the desired signal and $\dot{\theta}_{m2}(t) = \gamma_2 \sin \omega_{m2} t$ is the undesired signal, and if P_1 is the output signal power in the desired channel due to signal $\dot{\theta}_{m1}(t)$ and P_2 is the output power, due to $\dot{\theta}_{m2}(t)$, measured in the desired channel when the signal $\dot{\theta}_{m1}(t)$ is absent, then the intermodulation distortion(IMD) in the desired channel at frequency f_{m1} is given by

$$IMD(f_{m1}) = P_2/P_1 \qquad (8.3.8)$$

If the input signal $\dot{\theta}_m$ is a multiplexed FM signal consisting of many channels within the baseband, then the modulating signal is of the form given by

$$\dot{\theta}_m(t) = \sum_{k=1}^{I} \gamma_k \sin \omega_{mk} t \qquad (8.3.9)$$

where I is the number of channels in the baseband(practical systems have between 12 and 960 channels). When I is large, $\dot{\theta}_m$ may be considered as a band-limited Gaussian noise with zero mean. Then, the crosstalk in j^{th} channel is given by the ratio of the resultant signal power in channel j, due to all channels except channel j, to the signal power due to signal in channel j when signals in all other channels are absent.

A band limited Gaussian noise (with bandwidth equal to baseband frequency) may be generated and fed to the simulator (receiver) corresponding to the j^{th} channel and its output is transformed using FFT and let $|Y_{I-1}(\omega_{mj})|^2$ be the intermodulation power at the frequency f_{mj} due to all channels except j, then the crosstalk in channel j is given by

$$IMD(f_{mj}) = |Y_{I-1}(\omega_{mj})|^2/|Y_j(\omega_{mj})|^2 \tag{8.3.10}$$

where $|Y\ (\omega_{mj})|^2$ is the output power in the j^{th} channel due to the desired signal and is obtained from the spectrum of the output of the j^{th} simulator with input $\dot{\theta}_{mj} = \gamma_j \sin \omega_{mj} t$.

8.4 Volterra Series Characterisation of FM Discriminator and VCO Nonlinearities

In the previous section, it was assumed that the FM discriminator and voltage controlled oscillator are ideal and hence do not contribute nonlinear distortion. But, the deviation from linearity in the frequency detection characteristic of the FM discriminator, over the range of frequencies covered by the instantaneous frequency of its input signal, produces phase distortion. Further, the nonlinear input-output relationship, (i.e., amplitude -to-frequency conversion characteristic) of the voltage controlled oscillator over the range covered by its input voltage, produces additional phase shift which also results in phase distortion.

In this section, FM discriminator and VCO nonlinearities are represented by Volterra series and the corresponding nonlinear models obtained for distortion measurements.

8.4.1 Nonlinear Distortion due to FM Discriminator

The nonlinear distortion due to the deviation, from linearity, of the frequency-to-amplitude conversion characteristic of the FMD may be determined by representing its output $\dot{\rho}$ by a truncated power series in its input λ. Then, the discriminator output is given by

$$\dot{\rho}(t) = \dot{\lambda} + a\,\dot{\lambda}^2 + b\,\dot{\lambda}^3 \tag{8.4.1}$$

where a and b are coefficients of quadratic and cubic nonlinearities, respectively. In order to investigate the distortion due to discriminator only, it is assumed that the IF filter, loop-limiter and the VCO are ideal and do not contribute nonlinear distortion. Then, the system equations for the operation of the demodulator becomes

$$\begin{aligned} &\dot{y} = m_1 \otimes \dot{\rho} \quad , \quad \dot{\rho} = \dot{\lambda} + a\,\dot{\lambda}^2 + b\,\dot{\lambda}^3 \quad , \quad \dot{\lambda} = \dot{\delta} \quad , \\ &\dot{\delta} = k_1 \otimes \dot{\varepsilon} \quad , \quad \dot{\varepsilon} = \dot{\theta}_m - \dot{q} = \dot{\theta}_m - G\,\dot{y} \end{aligned} \tag{8.4.2}$$

where $\lambda = \delta$, since the loop-limiter is ideal and the output $\delta(t)$ of the IF filter is taken as linear in its input $\varepsilon(t)$.

Solving the system equations in transforms, using the linear system, multiplication and summing operations analogous to those described in Chapter 4, yields the kernels of the equivalent open-loop system L, as

$$\begin{aligned} L_1(s) &= \frac{M_1(s)\,K_1(s)}{\{1 + GM_1(s)K_1(s)\}} \\ L_2(s_1,s_2) &= \frac{aM_1(s_1+s_2)}{\{1+ GM_1(s_1+s_2)K_1(s_1+s_2)\}} \prod_{r=1}^{2} \frac{K_1(s_r)}{\{1+ GM_1(s_r)K_1(s_r)\}} \\ L_3(s_1,s_2,s_3) &= \frac{M_1(s_1+s_2+s_3)}{\{1+ GM_1(s_1+s_2+s_3)K_1(s_1+s_2+s_3)\}} \left[b \prod_{r=1}^{3} \frac{K_1(s_r)}{\{1+ GM_1(s_r)K_1(s_r)\}} \right. \\ &\quad \left. - \frac{2a\,GK_1(s_1)K_1(s_2+s_3)L_2(s_2,s_3)}{\{1+ GM_1(s_1)K_1(s_1)\}} \right] \end{aligned} \tag{8.4.3}$$

It may be observed that the linear kernel $L_1(s)$ is same as given by eqn. (8.2.15) and is shown in Fig.8.5(a). The second and third-order models, for measuring the distortion due to FMD, are shown in Figs.8.13(a) and (b), respectively. If $K_1(s)$ is a single-pole filter given by $K_1(s) = \frac{\alpha}{s+\alpha}$ and $M_1(s) = \frac{\omega(s+\alpha)}{\alpha(s+\omega)}$, the second and third-harmonic distortions may be determined by simulating the models and using the FFT, as in section8.3. The cross-talk in multiplexed FM signal may also be similarly obtained.

8.4.2 Distortion due to Nonlinear Voltage Controlled Oscillator

The nonlinearity in the amplitude-to-frequency conversion characteristic of the voltage controlled oscillator over the dynamic range covered by its input voltage will introduce phase distortion whose intensity increases with an increase in G. This nonlinear distortion may be evaluated by representing the VCO output q(t) by a truncated power series in its input $\dot{y}(t)$. Then, the VCO output becomes

$$q(t) = G\{y(t) + c\, y^2(t) + d\, y^3(t)\} \tag{8.4.4}$$

where c and d are the coefficients of second and third-order nonlinearities, respectively. Assuming that the IF filter, the loop-limiter and the FM discriminator do not contribute nonlinear distortion, the system equations for the operation of the demodulator may be written in the form

$$\begin{aligned} &\dot{y} = m_1 \otimes \dot{\rho} = m_1 \otimes \dot{\lambda} = m_1 \otimes \dot{\delta} \quad , \qquad \dot{\delta} = k_1 \otimes \dot{\varepsilon} \\ &\dot{\varepsilon} = \dot{\theta}_m - \dot{q} \quad , \qquad \dot{q} = G(\dot{y} + c\, \dot{y}^2 + d\, \dot{y}^3) \end{aligned} \tag{8.4.5}$$

where $\delta(t)$ is the distortion-free output of the IF filter and $\dot{\delta}(t)$ is the distortion-free output of the FM discriminator. If the frequency deviation is small, then the nonlinear distortion due to VCO may be negligible. But, if a sufficiently large deviation is used then the VCO nonlinearity will contribute significant distortion.

The system equations (8.4.5) may be solved using the linear system, summing and multiplication operations analogous to those described in Chapter 4 and the second and third-order kernels representing the equivalent open-loop system L may be obtained as

$$L_2(s_1,s_2) = \frac{-cG\, K_1(s_1+s_2)M_1(s_1+s_2)}{\{1+ GM_1(s_1+s_2)K_1(s_1+s_2)\}} \prod_{r=1}^{2} \frac{M_1(s_r)K_1(s_r)}{\{1+ GM_1(s_r)K_1(s_r)\}}$$

$$L_3(s_1,s_2,s_3) = \frac{-GM_1(s_1+s_2+s_3)K_1(s_1+s_2+s_3)}{\{1+ GM_1(s_1+s_2+s_3)K_1(s_1+s_2+s_3)\}} \left[d \prod_{r=1}^{3} \frac{M_1(s_r)K_1(s_r)}{\{1+ GM_1(s_r)K_1(s_r)\}} + \frac{2c\, M_1(s_1)K_1(s_1)L_2(s_2,s_3)}{\{1+ GM_1(s_1)K_1(s_1)\}} \right] \tag{8.4.6}$$

The linear kernel $L_1(s)$ is same as given by eqn.(8.2.15) which is shown in Fig.8.5(a). The second and third-order models for the measurement of distortion are shown in Fig.8.14(a) and (b), respectively. For $K_1(s)= \frac{\alpha}{s+\alpha}$ and $M_1(s) = \omega(s+\alpha)/\alpha(s+\omega)$, the variation of second and third-harmonic

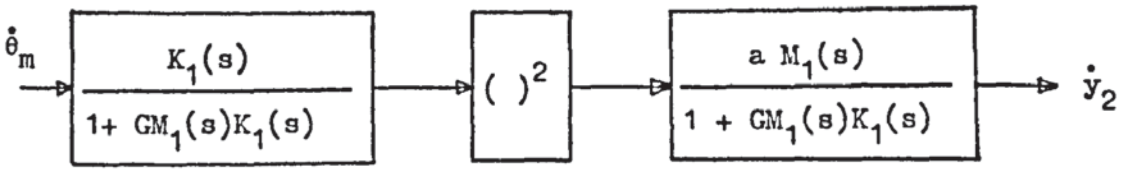


Fig.8.13(a) Second-order model for FMD distortion measurement.

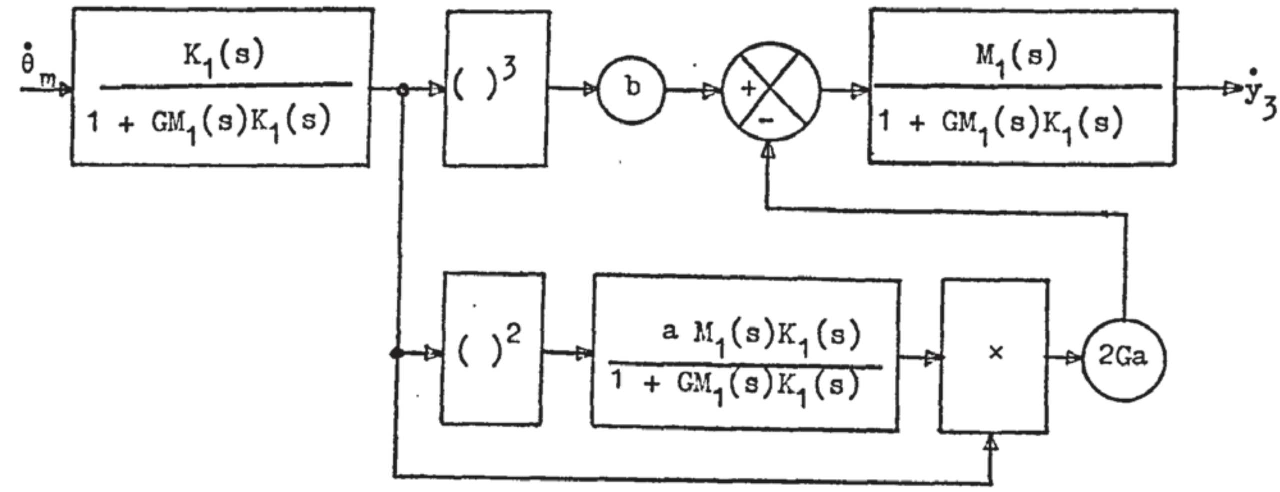


Fig.8.13(b) Third-order model for FMD distortion measurement.

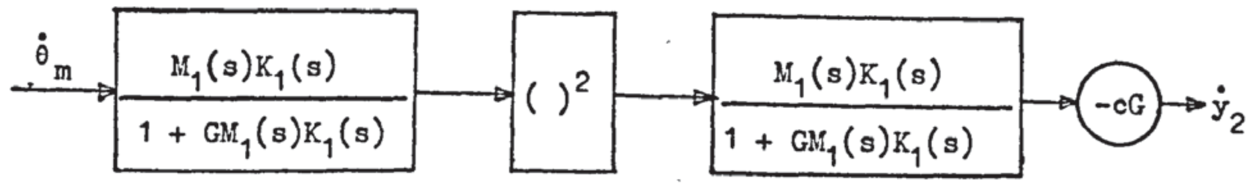


Fig.8.14(a) Second-order model for VCO distortion measurement.

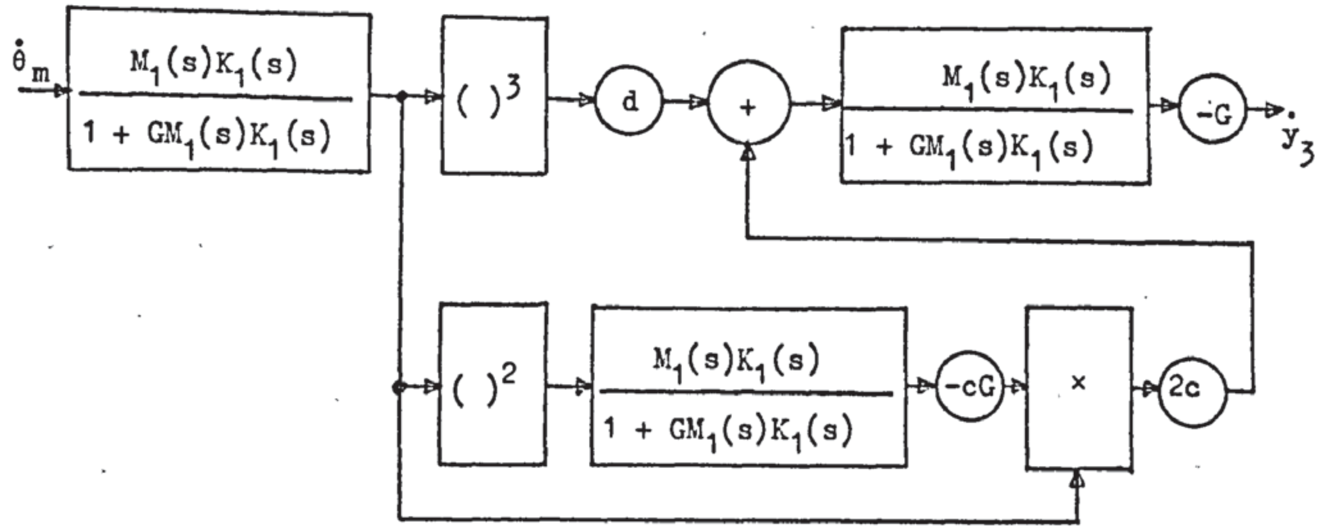


Fig.8.14(b) Third-order model for VCO distortion measurement.

distortion with various system parameters may be investigated as before. The models shown in Fig.8.14 are also useful in determining crosstalk in FDM/FM systems, using FBFM as receiver.

It is expected that[97], in general, the effect of the odd-order nonlinearities is to modify the frequency deviation by the fundamental components of the modulating signal, whereas the even-order nonlinearities cause a detuning of the centre frequency of the signal appearing at the input of the IF filter. These effects are in addition to introducing distortion.

8.5 Distortion Equaliser for FBFM Demodulator

The output $\dot{y}(t)$ consists of distortion terms in addition to the desired (linear) term, due to various nonlinearities in the demodulator. In certain wideband FDM/FM systems, that have stringent requirements on the allowable nonlinear distortion, it is desirable to have a distortion-free output. However, from the output $\dot{y}(t)$ of the demodulator, it is possible to obtain the desired distortion-free signal $\dot{\theta}_m(t)$ by passing $\dot{y}(t)$ through a "Distortion equaliser", which may be derived using functional inversion techniques[105]. In this section, a distortion equaliser is derived by expressing the input signal $\dot{\theta}_m(t)$, explicitly, in terms of the output signal $\dot{y}(t)$, using the inverse linear system operation.

The system equations (8.2.9),(8.4.2) and (8.4.5) may be combined to form the following system equations which represent the FBFM demodulator with nonlinearities in IF bandpass filter, loop-limiter, FM discriminator and voltage controlled oscillator.

$$\begin{aligned}
&\dot{y} = m_1 \otimes \dot{\rho} \quad , \qquad \dot{\rho} = \dot{\lambda} + a\dot{\lambda}^2 + b\,\dot{\lambda}^3 \,, \\
&\dot{\lambda} = f_1 \otimes \dot{\varepsilon} + C_p\, f_2 \otimes \dot{\varepsilon} + f_3 \otimes \dot{\varepsilon} \,, \qquad\qquad (8.5.1) \\
&\dot{\varepsilon} = \dot{\theta}_m - \dot{q} \quad , \qquad \dot{q} = G(\dot{y} + c\,\dot{y}^2 + d\,\dot{y}^3)
\end{aligned}$$

where $\dot{\rho}$ is the discriminator output, λ is the limiter output, δ is the IF filter output, q is the VCO output and $\dot{\theta}_m$, $\dot{\varepsilon}$ and $\dot{y}$ are the input, error and output signals, respectively. Then, an explicit expression for $\dot{\theta}_m$ in terms of $\dot{y}$ may be obtained as

$$
\begin{aligned}
\dot{\theta}_m &= \dot{\varepsilon} + \dot{q} \\
&= f_1^{-1} \odot (\dot{\lambda} - C_p\, f_2 \circledast \dot{\varepsilon} - f_3 \circledast \dot{\varepsilon}) + \dot{q} \\
&= k_1^{-1} \odot (\dot{\rho} - a\dot{\lambda}^2 - b\dot{\lambda}^3 - C_p\, f_2 \circledast \dot{\varepsilon} - f_3 \circledast \dot{\varepsilon}) + \dot{q} \qquad (8.5.2) \\
&= k_1^{-1} \odot (m_1^{-1} \odot \dot{y} - a\dot{\lambda}^2 - b\dot{\lambda}^3 - C_p\, f_2 \circledast \dot{\varepsilon} - f_3 \circledast \dot{\varepsilon}) + G(\dot{y} + c\dot{y}^2 + d\dot{y}^3)
\end{aligned}
$$

where the symbol $\odot$ denotes 'Inverse Linear System Operation', $k_1^{-1}(\tau)$ and $m_1^{-1}(\tau)$ are inverse impulse responses of lowpass (linear) filters. Since $K_1(s)$ and $M_1(s)$ are the transfer functions of $k_1(\tau)$ and $m_1(\tau)$, respectively, the inverse linear filters $k_1^{-1}(\tau)$ and $m_1^{-1}(\tau)$ will have transfer functions $1/K_1(s)$ and $1/M_1(s)$, respectively. Then, eqn.(8.5.2) represents the system equation for the operation of the distortion equaliser, which may be realised as shown in Fig.8.15. Since $k_1(\tau)$ and $m_1(\tau)$ are invertible, the feedback configuration of Fig.8.15 is stable. For $K_1(s)= \frac{\alpha}{s+\alpha}$ and $M_1(s)= \frac{\omega(s+\alpha)}{\alpha(s+\omega)}$, the equaliser may be designed and when this equaliser is placed at the output of the demodulator, the nonlinear distortion in the demodulated output is reduced and the desired distortion-free signal $\dot{\theta}_m(t)$ is obtained.

8.6 Conclusions

It has been shown that the use of the Volterra series and the transform methods for the analysis of nonlinear distortion in FBFM demodulator yields lower values of distortion than those obtained by others. Various nonlinear models have been derived for precise measurement of distortion due to each of the nonlinearities contributed by IF filter, loop-limiter, FM discriminator and voltage controlled oscillator. Optimum values of system parameters have also been suggested for obtaining low distortion and for properly designing the demodulator.

Further reduction in the distortion may be achieved by increasing the feedback gain, which introduces severe practical problems in such a wideband circuitry, or by increasing the bandwidth of the IF filter, which reduces the threshold-extension capability of the demodulator. However, an alternative method for reducing the nonlinear distortion in the demodulated output is to use the distortion equaliser, derived here, at the output of the demodulator, which provides the distortion-free signal $\dot{\theta}_m(t)$.

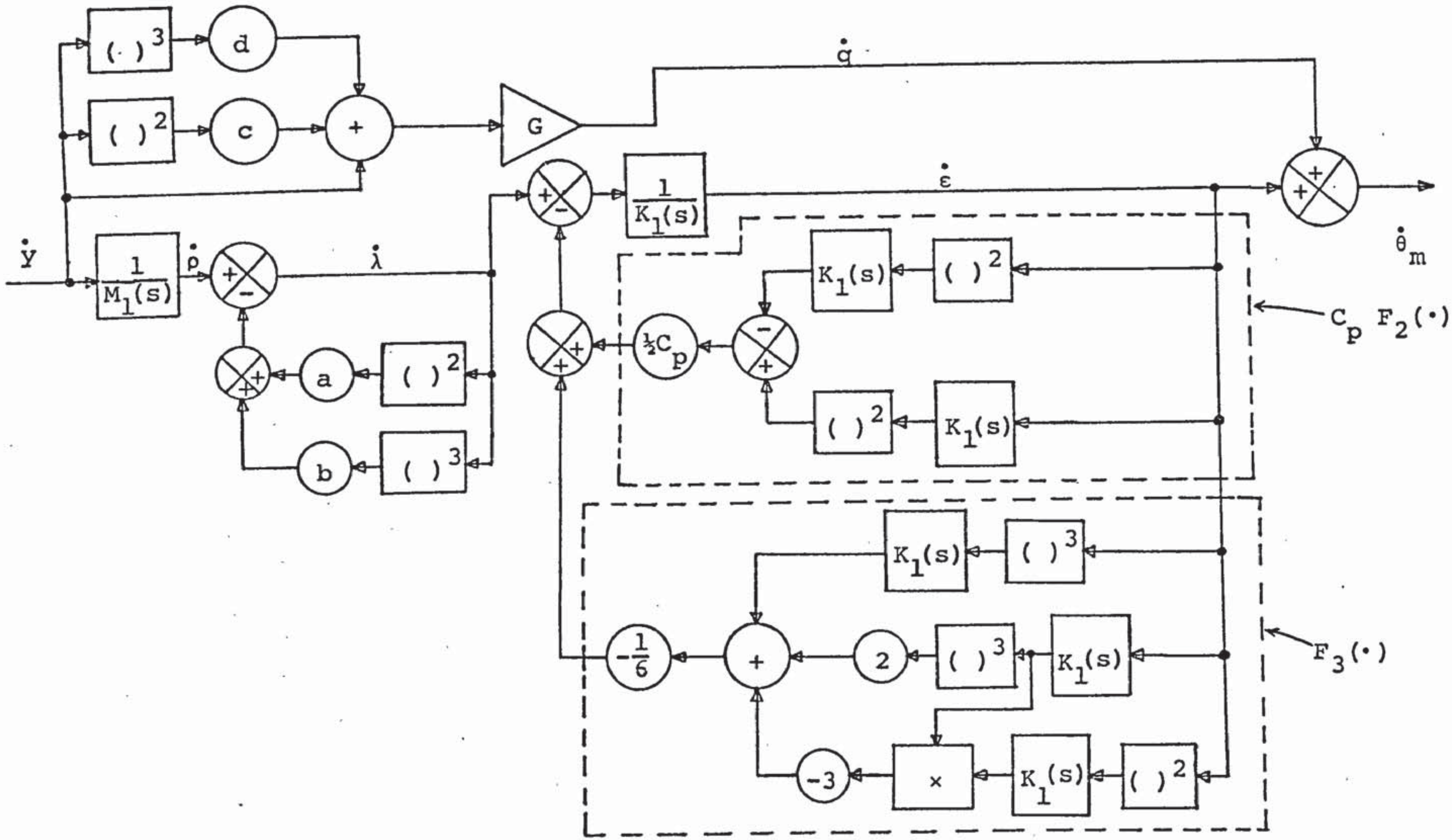


Fig.8.15. Distortion equaliser for FBFM demodulator.

CHAPTER 9

CONCLUSIONS

9.1 Discussion

The analysis of sampled-data nonlinear systems by transform methods has, in the past, been restricted to relatively simple cases because of the difficulty in obtaining the multidimensional z transform of a nonlinear system kernel. This difficulty has been overcome by the method developed here, in which the multidimensional z transform of a nonlinear system kernel is obtained by a sequential process applied to the multidimensional Laplace transform of the kernel, which is easily synthesised for a large class of nonlinear systems. The procedure at each stage of this process is a simple one, which, in many cases, may be carried out by inspection or the calculation of residues.

The sequential process also allows the multidimensional z transform of a nonlinear system cascaded with a data-hold device, commonly associated with sampled-data systems, to be obtained from the derived multidimensional Laplace transform of the cascade. The most frequently used data-hold devices are the zero-order hold, which gives an exact solution to stepwise continuous signal and the first-order hold, which gives an exact solution to rampwise continuous signal. Both these devices could be handled by the sequential process depending on the nature of the input signal. It has been observed that, for nonlinear systems with fourth and higher order kernels cascaded with a data-hold device, the application of the sequential process to obtain the multidimensional z transform of the cascade leads to tedious algebraic manipulations. But, in many cases of practical interest, kernels of order four and higher may be neglected since most of the physical systems may be adequately characterised by the first three Volterra kernels and the Volterra series solution, when truncated after a finite number of terms, provides a fairly reasonable approximation to the exact solution.

The sampled-data output of a nonlinear system with a given sampled-data input, can be obtained by the sequential process based on or the theorems of, the association-of-variables procedure developed here; this also leads to a simple procedure which, in many cases, may be carried out by inspection or the calculation of residues. For the analysis of sampled-data systems with asynchronous input-output sampling, similar procedures based on multidimensional modified z transforms may be used. It should be noted that if two subsystems are separated by a sampler, then the multidimensional modified z transform of the cascade is given by the product of the multidimensional modified z transform of the second subsystem and the multidimensional z transform of the first subsystem, as in the linear case, because the output of the second subsystem is modified whereas the output of the first subsystem is not modified.

The theory of operator algebra has been extended to nonlinear discrete systems in which zero-order subsystems are assumed to be present. The explicit input-output relationships developed for six basic operations can be used for the analysis of sampled-data nonlinear feedback systems, in which the component subsystems may or may not be separated by a sampler.

The method of obtaining the sampled-data output of a given nonlinear system, based on association-of-variables procedure becomes cumbersome for inputs other than a sampled step. Further, this procedure has to be carried out each time when the input signal is changed. This difficulty has been overcome by the synthesis procedure developed here, by which the multidimensional z transform of each of the kernels, characterising the given system, may be synthesised using a finite number of multipliers and first and second-order linear discrete systems. This yields a discrete simulator for each of the kernels cascaded with a data-hold device. Then, the response of the system for any input signal may be obtained by applying the input to each of the simulators and adding the outputs of all the simulators. These discrete simulators are also quite useful for dynamic system investigations such as measurement of weighting function by cross-correlation or system frequency response by spectral techniques, using pseudorandom input signals.

The state variable analysis, hitherto confined to linear systems, has been extended to nonlinear single-variable and multivariable systems. It has been shown how the nonlinear systems with multiplicative, functional or polynomial type nonlinearities may be characterised in state space by means of the dynamic equations. The solution of the dynamic equations in all cases shows explicitly the influence of the state transition matrix, which operates on each of the terms of the Volterra series expansion to obtain the state and the output of the system in terms of the initial conditions and the input signal. The explicit input-output relationship of a multi-input, multi-output system in terms of its transition matrix has been established, by which the multidimensional Laplace transform kernels characterising such a system may be derived and synthesised. The utility of the method may be seen from the illustrative example given in chapter 6, in which the response and the multidimensional Laplace transform kernels of a two-input, single-output diode ring multiplier are obtained.

Two methods of solution have been developed in chapter 7 for the state variable description of sampled-data nonlinear systems with their inputs applied through data hold devices: one for the single-input single-output system and the other for the multivariable system. The former method of solution does not give an explicit input-output relationship while the latter method developed for multi-input, multi-output sampled-data system gives an explicit input-output relationship in terms of the discrete state transition matrix. This relationship permits the multi-dimensional z transform kernels characterising such a system to be derived and synthesised in terms of the discrete state transition matrix, thereby providing a general method for the digital simulation of a multivariable nonlinear system with inputs applied through sample-and-hold devices. Thus, the state space solution of nonlinear systems using multidimensional transform methods parallels the state space solution of linear systems using one-dimensional transform methods.

The second and third-order models of a practical feedback FM demodulator have been derived, by the Volterra series and the multidimensional transform methods, for the measurement of distortion and crosstalk in the received FM signal. The effects of various system parameters on the distortion have also been studied and optimum design parameters suggested. The results obtained in this thesis are found to be better than those obtained by others. A distortion equaliser has been derived using the theory of inverse functionals, which, when connected to the output of the demodulator, eliminates the nonlinear distortion in the demodulated output and provides a distortion-free message signal. This amply shows the utility of Volterra functional series in nonlinear modelling of physical systems.

9.2 Advantages and Disadvantages

The main advantages of these methods of analysis and synthesis are:

(a) Nonlinear differential or difference equations, characterising a given system, when transformed into multidimensional Laplace or z domain, respectively, are reduced to algebraic equations, which may be easily handled.

(b) The procedure for obtaining the Volterra series solution of the output of a given system is fairly easy, which, in many cases, may be carried out by inspection or calculation of residues.

(c) The form of the solution provided by this method of analysis is such that the overall response of the system may be viewed as the superposition of responses of the linear kernel, second-order kernel etc., and hence it would enable one to assess the effect of nonlinearities in the system on the overall system response. This superposition property of the Volterra series also enables one to determine the characteristics of higher-order kernels of an unknown system by higher dimensional cross-correlation of the input and the output of the system, provided that the input signal has Gaussian noise properties.

(d) It gives transient as well as steady state response of the system, to any arbitrary input, in which the system parameters appear explicitly thereby providing considerable insight into the behaviour of the system.

(e) The discrete simulators of a given nonlinear system with kernels cascaded with data-hold devices, are general and are valid for any input signal, which make them quite useful for dynamic system investigations such as correlation testing and spectral analysis.

However, the only drawback with this method of analysis and synthesis is that when the system nonlinearity is not small, more number of terms in the Volterra series solution are required for better convergence to the exact solution and this, in general, leads to the determination of multidimensional z transforms of higher-order kernels cascaded with a data-hold device, which involves tedious and lengthy calculations. However, when the system nonlinearity is not violent, the truncated Volterra series solution converges rapidly and fastly to the exact solution. This may be observed from the illustrative example given in chapter 5, in which the response of a system, whose dynamics is dependent on the direction of the input signal, has been obtained for both positive and negative step inputs.

9.3 Applications and Suggestions for Further Work

The applications of the results presented in this thesis include the following fields: analysis of complex nonlinear systems(open-loop or feedback) with or without samplers between various subsystems; digital simulation of continuous nonlinear systems cascaded with data-hold devices, required for dynamic system investigations such as correlation testing and spectral analysis; computer control systems; and nonlinear modelling and simulation of physical systems.

The areas of further development of this project include: study of the stability and convergence properties of discrete Volterra series using transform methods; optimum control of nonlinear systems using state variable feedback; nonlinear distributed parameter systems; and a general and simplified method of obtaining a canonic discrete simulator for a nonlinear system cascaded with a data-hold device.

REFERENCES

1. Gillies,A.W.:'The application of power series to the solution of non-linear circuit problems', Proc. IEE, Vol.96, part III, 1949, pp.453-75.
2. Volterra,V.:'Theory of functionals and of integral and integro-differential equations', Dover Publications, Inc., N.Y., 1959
3. Wiener,N.:'Differential space', J.Math and Phys.,2, 1923, pp.131-174
4. Wiener,N.:'Nonlinear problems in random theory', The Technology Press, Cambridge, Mass., and John Wiley, N.Y., 1958
5. Wiener,N.:'Response of a nonlinear device to noise', Radiation Laboratory report V-16S, MIT, Cambridge, Mass., April 1942
6. Bose,A.G.:'A theory of nonlinear systems', Tech. report 309, MIT Research Lab. of Electronics, Cambridge, Mass., May 1956
7. Chesler,D.A.:'Nonlinear systems with gaussian inputs', Tech. report 366, MIT Research Lab. of Electronics, Cambridge, Mass., Feb. 1960
8. Schetzen,M.:'Some problems in nonlinear theory', Tech. report 390, MIT Research Lab. of Electronics, Cambridge, Mass., July 1962
9. Van Trees,H.L.:'Synthesis of optimum nonlinear control systems', Research monograph, MIT Press, Cambridge, Mass., 1962
10. Reddy,P.K. and Reddy,D.C.:'Volterra kernels in nonlinear systems', Electronics Letters, Vol. 9, No. 18, 1973, pp. 426-27.
11. Barrett,J.F.:'The use of functionals in the analysis of nonlinear physical systems', Journal of Electronics and Control, Vol. 15, No.6, Dec. 1963, pp.567-615
12. Brilliant,M.B.:'Theory of the analysis of nonlinear systems', Tech. report 345, MIT Research Lab.of Electronics, Cambridge, Mass., Mar.1958
13. George,D.A.:'Continuous nonlinear systems', Tech. report 355, MIT Research Lab. of Electronics, Cambridge, Mass., July 1959
14. Zames,G.:'Nonlinear operators for system analysis', Tech. report 370, MIT Research Lab. of Electronics, Cambridge, Mass., Aug. 1960
15. McFee,R.:'Determining the response of nonlinear systems to arbitrary inputs', Trans. Amer. Instn. Elec. Engrs., Vol.80, Sept.1961, pp.189-93

16. Flake,R.H.: 'Volterra series representation of nonlinear systems', Trans. Amer. Instn. Elect. Engrs., No. 64, Jan. 1963, pp.330-335

17. Bansal,V.S.: 'Volterra functional analysis of nonlinear systems', Ph.D. Thesis, University of Glasgow, U.K., 1970

18. Lubbock,J.K.: 'Nonlinear filters', Progress in Radio Science 1960-63, Vol. VI, Elsevier Publishing Co., Amsterdam, 1966, pp.127- 166.

19. Halme,A., Orava,J. and Blomberg,H.: 'Polynomial operators in nonlinear systems theory', Int. J of Syst. Sci., Vol. 2, No. 1, 1971, pp.25-47

20. Flake,R.H.: 'Volterra series representation of time-varying nonlinear systems', Proc. of the second Congress of IFAC, Basle, Switzerland, 1963, pp.91-99

21. Ku,Y.H. and Su,C.C.: 'Volterra functional analysis of nonlinear time-varying systems', Journal of Franklin Inst., Vol.284, 1967, pp.344-365

22. Ridings,R.V. and Higgins,T.J.: 'Transient response analysis of a class of continuous nonlinear time-varying automatic control systems by functional techniques and multidimensional Laplace transforms', ISA Trans., Vol. 7, No. 2, 1968, pp. 166-172.

23. Bansal,V.S.: 'Volterra-series analysis of a class of nonlinear time-varying systems using multilinear parametric transfer functions', Proc. IEE, (Correspondance), Vol. 116, No. 11, 1969, pp. 1957-1960

24. Alper,P.: 'Some aspects of the Volterra series', Tech. report, Electronics Lab., Technological University, Delft, Holland, 1964

25. Barrett,J.F.: 'The use of Volterra series to find the region of stability of a nonlinear differential equation', Int. J of Control, Vol. I, No.3, March 1965, pp. 209-216

26. Ku,Y.H. and Wolf,A.A.: 'Volterra-Wiener functionals for the analysis of nonlinear systems', Journal of Franklin Inst., Vol.281, 1966, pp.9-26

27. Parante,R.B.: 'Functional analysis of systems characterised by nonlinear differential equations', Tech. report 444, MIT Research Lab. of Electronics , Cambridge, Mass., July 1966

28. Marchesini,G.: 'On a bounded input-bounded output sufficient condition

for nonlinear systems', IEEE Trans. on Automatic Control, Vol. AC-13, June 1968, pp. 305-307

29. Christensen,G.S.: 'On the convergence of Volterra series', ibid., 736-37

30. Trott,G.W. and Christensen,G.S.: 'On the uniqueness of the Volterra series', ibid., Vol. AC-14, Dec 1969, pp. 759-760

31. Christensen,G.S. and Trott,G.W.: 'On the inclusion of initial conditions in Volterra series', Int. J of Control, Vol.12, No.5, 1970, pp.835-838

32. Trott,G.W. and Christensen,G.S.: 'A larger region of convergence for the Volterra series', ibid., Vol.14, No. 2, 1971, pp. 377-384

33. Anjaiah,A. and Reddy,D.C.: 'Equivalence of multidimensional transforms and reversion methods of solving nonlinear equations', Proc. IEE, Vol. 120, No. 1, Jan 1973, pp. 144-145

34. Widnall,W.S.: 'Measurement of second-degree Wiener kernel in a nonlinear system by cross-correlation', MIT QPR 65, Research Lab. of Electronics, Cambridge, Mass., April 1962, pp. 132-143

35. Lee,Y.W. and Schetzen,M.: 'Measurement of the Wiener kernels of a non-linear system by cross-correlation', Int. J of Control, Vol.II, No. 3, Sept. 1965, pp. 237-254

36. Huang,T.: ' Measurement of Volterra kernels of a nonlinear system of a finite order', M.Sc. Thesis, MIT RLE, Cambridge, Mass., Aug. 1965

37. Schetzen,M.: 'Measurement of the kernels of a nonlinear system of finite order', Int. J of Control, Vol. I, No. 3, March 1965, pp. 251-263

38. Hooper,R.J. and Gyftopoulos,E.P.: 'On the measurement of characteristic kernels of a class of nonlinear systems', Presented at the Symposium on Neutron Noise, etc., University of Florida, Fla., U.S.A., Feb. 1966

39. Gyftopoulos,E.P. and Hooper,R.J.: 'Signals for transfer-function measurement in nonlinear systems', in 'Noise analysis in nuclear systems', USAEC Symposium series 4, TID- 7679, 1964

40. Barker,H.A. and Pradisthayon,T.: 'High-order autocorrelation functions of pseudorandom signals based on m sequences', Proc. IEE, Vol. 117, No. 9, pp. 1857-1863, 1970

41. Barker,H.A. and Obidegwu,S.N: 'Combined crosscorrelation method for the measurement of second order Volterra kernels', Proc. IEE, Vol.120, No.1, Jan 1973, pp. 114-118

42. Barker,H.A. and Obidegwu,S.N: 'Effects of nonlinearities on the measurement of weighting functions by crosscorrelation using pseudorandom signals' , Proc. IEE, Vol. 120, No. 10, Oct.1973, pp. 1293-1300

43. Sinha,A.K. and Mahalanabis,A.K.: 'Recursive identification of linear and nonlinear systems', Int. J of Syst. Sci., Vol.5, No.11,1974,pp.1065-1076

44. Gardiner,A.B.: 'Frequency domain identification of nonlinear systems', Proc. of IFAC Symposium, Hague, Netherlands, 1973

45. Barker,H.A. and Davy,R.W.: 'System identification using pseudorandom signals and the discrete Fourier transform', Proc. IEE, Vol.122, No.3, March 1975, pp. 305-311

46. Contos,J.: 'Simulation studies in respiratory mechanics and magnetohydrodynamics', Ph.D. thesis, Univ. of Birmingham, England, 1963

47. Zames,G.: 'Functional analysis applied to nonlinear feedback systems', IEEE Trans. on Circuit Theory, Vol. CT-10, Sept. 1963, pp. 392-404

48. Berger,J.B.: 'Application of Volterra series analysis to an eletrohydraulic control valve', Joint Automatic Control Conference,1964,pp.62-69

49. Van Trees,H.L.: 'Functional techniques for the analysis of the nonlinear behaviour of phase-locked loops', IEEE Proc.,Vol.52,1964,pp. 894-911

50. Waddington,J. and Fallside,F.: 'Analysis of nonlinear equations by the Volterra series', Int. J of Control,Vol.3, No.1, 1966, pp. 1-15

51. Lavi,A. and Mastascusa,E.J.: 'Analysis of an extremum seeking nonlinear system', Journal of Franklin Inst., Vol. 281, 1966, pp. 235-249

52. Roy,R.J. and Sherman,J.: 'A learning technique for Volterra series representation', IEEE Trans. Automatic Control, Vol.AC-12, 1967,pp.761-64

53. Narayanan,S.: 'Transistor distortion analysis using Volterra series representation', The Bell Syst. Tech. Journal,Vol.46, 1967, pp.991-1024

54. Narayanan,S.: 'Application of Volterra series to intermodulation distortion analysis of transistor feedback amplifiers', IEEE Trans. on Circuit

theory, Vol. CT-17, No.4, Nov. 1970, pp. 518-527

55. Narayanan,S.: 'Intermodulation distortion of cascaded transistors', IEEE Jour. of Solid-State Circuits, Vol.SC-4, No.3, June 1969, pp. 97-106

56. Maurer,R.E. and Narayanan,S.: 'Noise loading analysis of a third-order nonlinear system with memory', IEEE Trans. on Communication Tech., Vol. COM-16, No. 5, Oct. 1968, pp. 701-712

57. Stark,L.: 'The pupillary control system: its nonlinear adaptive and stochastic engineering design characteristics', Automatica, Vol.5, 1969, pp. 655-676

58. Bansal,V.S.: 'Study of boundedness of the response of a nuclear reactor using Volterra series', Proc. IEE, Vol.116, No.12, Dec.1969,pp.2072-74

59. Neill,T.B.M.: 'Improved method of analysing nonlinear electrical networks', Electronics Letters, Vol. 5, No. 1, 1969, pp. 13-15

60. Goldman,J.: 'A Volterra series description of crosstalk interference in communications systems', The Bell Syst. Tech. Journal,Vol.52, 1973, No. 5, pp. 649-668

61. Narayanan,S. and Poon,H.C.: 'An analysis of distortion in Bipolar transistors using integrated charge control model and Volterra series', IEEE Trans. on Circuit Theory, Vol. CT-20, No. 4, 1973, pp.341-351

62. Rice,S.O.: 'Volterra systems with more than one input port-distortion in a frequency converter', The Bell Syst. Tech. Journal, Vol. 52, No. 8, Oct. 1973, pp. 1255-1270

63. Meyer,R.G. and Stephens,M.L.: 'Distortion in variable-capacitance diodes', IEEE Jour. of Solid-State Circuits, Vol.SC-10, No.1, 1975, pp.47-54

64. Lawless,W.J. and Schwartz,M.: 'Binary signaling over channels containing quadratic nonlinearities', IEEE Trans. on Communication Tech., Vol. COM-22, No. 3, 1974, pp.288-297

65. Landau,M. and Leondes,C.T.: 'Volterra series synthesis of nonlinear stochastic tracking systems', IEEE Trans. Aerospace and Electronic systems, Vol. AES-11, No.2, 1975, pp.245-265

66. Kuo,B.C.: 'Analysis and synthesis of sampled-data control systems',

Printice-Hall, Inc., Englewood Cliffs, N.J.,1963

67. Kalman,R.E. and Bertram,J.E.: 'Control systems analysis and design via second method of Liapunov II discrete-time systems', Trans. A.S.M.E., June 1960

68. Bush,A.M.: 'Transform analysis of nonlinear systems with sampled input and output', MIT QPR 75, Research Lab. of Electronics, Cambridge, 1964

69. Alper,P.: 'A consideration of the discrete Volterra series', IEEE Trans., on Automatic Control, Vol.AC-10, July 1965, pp. 322-327

70. Lavi,A. and Narayanan,S.: 'Analysis of a class of nonlinear discrete system using multidimensional modified z-transforms', IEEE Trans. on Automatic Control, Vol.AC-13, Feb. 1968, pp. 90-93

71. Barker,H.A.: 'Synchronous sampling theorem for nonlinear systems', Electronics Letters, Vol.5, No. 25, 1969, pp. 657

72. Jagan,N.C. and Reddy,D.C.: 'Evaluation of response of nonlinear discrete systems using multidimensional z transforms', Proc. IEE, Vol.119, No.10, Oct. 1972, pp. 1521-1525

73. Lubbock,J.K. and Bansal,V.S.: 'Multidimensional Laplace transforms for the solution of nonlinear equations', Proc. IEE, Vol.116, No.12,1969, pp. 2075-2082

74. Fu,F.C. and Farison,J.B.: 'Analysis of a class of nonlinear discrete-time systems by the Volterra series', Int. J of Control, 1973, Vol. 18 No. 3, pp. 545-551

75. Fu,F.C. and Farison,J.B.: 'On the Volterra series functional evaluation of the response of nonlinear discrete-time systems', ibid., pp.553-558

76. Marchesini,G. and Picci,G.: 'On the functional identification of non-linear systems from input-output data records', IEEE Trans. Automatic Control, Vol. AC-14, 1969, pp.757-759

77. Padgett,W.J. and Tsokos,C.P.: 'A random discrete equation and nonlinear stochastic control system', Int. J of Syst. Sci., Vol.4,1973, pp.77-85

78. Bansal,V.S. and Mali,S.L.: 'Analysis of a class of nonlinear time-varying discrete systems using multidimensional parametric z transfer

functions', Proc. IEE, Vol. 121, 1974.

79. Jagan,N.C. and Desai,R.C.: 'Analysis of nonlinear time-varying discrete systems by the use of multidimensional z transforms', Proc. IEE, Vol. 121, No. 4, April 1974, pp. 307-309

80. Rao,R.S. and Christensen,G.S.: 'On the convergence of a discrete Volterra series', IEEE Trans. on Automatic Control, Vol.AC-15, 1970, pp.140-141

81. Rashed,A.M.H. and Christensen,G.S.: 'Response of nonlinear sampled-data systems with nonzero initial conditions and response for in-between sampling via Volterra series', IEEE Trans. on Automatic Control, Vol. AC-16, 1971, pp. 269-270

82. Alper,P.: 'Higher dimensional z transforms and nonlinear discrete systems', Electronics Laboratory, Dept. of Electrical Engineering, Technological University, Delft, Netherlands, Sept. 1963

83. Kuo,B.C.: 'Discrete-data control systems', Prentice-Hall, Inc., Englewood Cliffs, New Jersey, 1970

84. Reddy,D.C. and Jagan,N.C.: 'Multidimensional transforms: new technique for the association of variables', Electronics Letters, Vol.7, 1971, pp. 278-279

85. Chen,C.F. and Chiu,R.F.: 'New theorems of association of variables in multiple dimensional Laplace transform', Int. J of Syst. Sci., Vol.4, No. 4, 1973, pp.647-664

86. Koh,E.L.: 'Association of variables in n-dimensional Laplace transform', Int. J of Syst. Sci., Vol. 6, 1975, pp.127-131

87. Barker,H.A.: 'Nonlinear systems analysis by multidimensional transform methods', Lecture presented to I.E.E., London, May 1973.

88. Bush,A.M.: 'Some techniques for the synthesis of nonlinear systems', MIT Research Laboratory of Electronics, Report 441, Cambridege, Mass., 1966

89. Obidegwu,S.N: 'Identification of nonlinear systems by correlation using pseudorandom signals', Ph.D thesis, Univ. of Aston in Birmingham,U.K.,1974

90. Schetzen,M: 'Synthesis of a class of nonlinear systems', Int. J of Control, Vol. 1, No.5, 1965, pp. 401-414

91. Bansal,V.S. and Onkar Goyal.: 'Synthesis of a class of nonlinear systems in multidimensional frequency domain', Proc.IEE, Vol. 121, 1974, pp 883-85

92. Godfrey,K.R and Briggs,P.A.N.: 'Identification of processes with direction dependent dynamic responses', Proc.IEE., Vol. 119, 1972, pp. 1733-1739

93. Ferguson,D.R.: 'General power series description of multiplicative non-linear systems',Electronic Letters, Vol. 5, No. 19, 1969, pp. 450-52

94. Wylie,C.R.: 'Advanced Engineering Mathematics', ch. 13, Third Edition, Mcgraw Hill Book Company, N.Y., U.S.A., 1966

95. Narayanan,S.: 'Transform methods for special nonlinear systems', Ph.D dissertation, Carnegie-Mellon University, U.S.A., 1965.

96. Chaffee,J.G.: 'Application of negative feedback to frequency-modulated systems', Bell System Technical Journal, Vol. 18, 1939.

97. Baghdady,E.J.: 'The theory of FM demodulation with frequency-compressive feedback', IRE Trans. Communication Systems, Vol. CS-10, 1961, pp. 226-45

98. Taub,H. and Schilling,D.L.: 'Principles of communication systems', Mcgraw Hill Book Company, N.Y., U.S.A., 1971

99. Hoffman,E. and Schilling,D.L.: 'Distortion in the frequency demodulators using feedback', IEEE Trans. on Commns., Vol. COM-20, 1972, pp. 157-165

100. Singleton,R.C.: 'An algorithm for computing the mixed radix fast Fourier transform', IEEE Trans. on Audio & Electroacoustics, Vol.17, 1969, pp.93-103

101. Nelin,B.: 'Baseband modelling and distortion equalisation of the feedback FM demodulator by functional methods', IEEE Trans. Aerospace and Electronic Systems, Vol. AES-10, 1974, pp. 149-152

102. Panter,P.F.: 'Communication Systems Design', Mcgraw Hill Book Company, Inc., New York, U.S.A., 1972, ch. 7, pp. 190-193

103. Bedrosian,E. and Rice,S.O.: 'Distortion and crosstalk of linearly filtered, angle modulated signals', Proc. IEEE, Vol.56, 1968, pp. 2-14

104. Mircea,A.: 'Harmonic distortion and intermodulation noise in linear FM transmission systems', Rev. Roum. Sci. Tech., Vol.12, Nov. 3, 1967

105. Frank,P. and McFee,R.: 'Determining input-output relationships of non-linear systems with inversions', IEEE Trans. Vol. CT-10, 1963, pp. 168-180

APPENDIX A.3

A.3.1 Properties and Theorems of Multidimensional Modified Z Transforms

Some of the properties of multidimensional modified z transforms, (M.D.M.Z.T), which may be useful in predicting the system's response behaviour in time domain, from the available transform domain information, are discussed here. The transforms considered here are unilateral unless stated otherwise. The properties of the multidimensional z transform(M.D.Z.T) may be obtained from the corresponding properties of M.D.M.Z.T by letting m=0.

A.3.1.1 Initial Value Theorem

If, $F_n(m,z_1,z_2,\ldots,z_n)$ is the M.D.M.Z.T of a causal function $f_n(k_1T, k_2T,\ldots,k_nT)$, for $0 \leq m < 1$, then the initial value of $f_1(kT)$ is given by

$$f_1(0) = \lim_{\substack{k_r\to 0,\\ 1\leq r\leq n}} f_n(k_1T,k_2T,\ldots,k_nT) = \lim_{\substack{z_r\to\infty\\ 1\leq r\leq n}} F_n(m,z_1,z_2,\ldots,z_n)\Big|_{m=0} \quad (A.3.1)$$

Proof: For causal function $f_n(t_1,t_2,\ldots,t_n)$, one has, by definition,

$$F_n(m,z_1,z_2,\ldots,z_n) = \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} f_n(<k_1+m>T,<k_2+m>T,\ldots,<k_n+m>T) \times \prod_{r=1}^{n} z_r^{-k_r} \quad (A.3.2)$$

Taking the limit, inside the summation, as $z_1\to\infty$ gives

$$\lim_{z_1\to\infty} F_n(m,z_1,z_2,\ldots,z_n)$$

$$= \sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} \lim_{z_1\to\infty} f_n(<k_1+m>T,<k_2+m>T,\ldots,<k_n+m>T)\prod_{r=1}^{n} z_r^{-k_r}$$

$$= \sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} f_n(mT,<k_2+m>T,\ldots,<k_n+m>T)\prod_{r=2}^{n} z_r^{-k_r}$$

Similarly, taking the limit as $z_2\to\infty$, $z_3\to\infty$, ..., and as $z_n\to\infty$ and letting m=0 yields eqn.(A.3.1), as required.

A.3.1.2 Real Translation Theorem

This theorem has two parts: (a) Backward shifting theorem and (b) Forward shifting theorem.

(a) Backward Shifting Theorem

If $F_n(m,z_1,z_2,\ldots,z_n)$ is the M.D.M.Z.T of $f_n(t_1,t_2,\ldots,t_n)$, then

$$\mathcal{Z}\left[f_n(<k_1+m-i_1>T,<k_2+m-i_2>T,\ldots,<k_n+m-i_n>T)\right]$$

$$= \left(\prod_{r=1}^{n} z_r^{-i_r}\right) F_n(m, z_1, z_2, \ldots, z_n) \quad , \quad 0 \leq m < 1 \tag{A.3.3}$$

Proof: By definition,

$$\mathcal{Z}\left[f_n(\langle k_1+m-i_1\rangle T, \langle k_2+m-i_2\rangle T, \ldots, \langle k_n+m-i_n\rangle T)\right]$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f_n(\langle k_1+m-i_1\rangle T, \langle k_2+m-i_2\rangle T, \ldots, \langle k_n+m-i_n\rangle T) \prod_{r=1}^{n} z_r^{-k_r}$$

Noting that $f_n(k_1T, k_2T, \ldots, k_nT) = 0$ for any $k_r < 0$, $r=1,2,\ldots,n$, the above equation may be written as

$$\mathcal{Z}\left[f_n(\langle k_1+m-i_1\rangle T, \langle k_2+m-i_2\rangle T, \ldots, \langle k_n+m-i_n\rangle T)\right]$$

$$= \left(\prod_{r=1}^{n} z_r^{-i_r}\right) \sum_{k_1=i_1}^{\infty} \sum_{k_2=i_2}^{\infty} \cdots \sum_{k_n=i_n}^{\infty} f_n(\langle k_1+m-i_1\rangle T, \langle k_2+m-i_2\rangle T, \ldots, \langle k_n+m-i_n\rangle T)$$
$$\times \prod_{r=1}^{n} z_r^{-(k_r-i_r)}$$

Letting $(k_r-i_r) = j_r$, for $r=1,2,\ldots,n$, yields eqn.(A.3.3), as required.

Corollary: If $F_n(m, z_1, z_2, \ldots, z_n)$ is the M.D.M.Z.T of $f_n(t_1, t_2, \ldots, t_n)$, then, for shifts not equal to an integral number of sampling intervals[88],

$$\mathcal{Z}\left[f_n(\langle k_1+m-\Delta\rangle T, \langle k_2+m-\Delta\rangle T, \ldots, \langle k_n+m-\Delta\rangle T)\right]$$

$$= \left(\prod_{r=1}^{n} z_r^{-1}\right) F_n(\{m+1-\Delta\}, z_1, z_2, \ldots, z_n) \quad , \quad 0 \leq m < \Delta < 1$$

$$= F_n\left[(m-\Delta), z_1, z_2, \ldots, z_n\right] \quad , \quad 0 \leq \Delta \leq m < 1 \tag{A.3.4}$$

(b) Forward Shifting Theorem

The general expression for forward shifting in n-dimensional case is, rather, complicated. Hence, only the two dimensional case is considered here. If $F_2(m, z_1, z_2)$ is the M.D.M.Z.T of $f_2(t_1, t_2)$, then

$$\mathcal{Z}\left[f_2(\langle k_1+m+1\rangle T, \langle k_2+m+1\rangle T)\right] = z_1 z_2 \{F_2(m, z_1, z_2) - \lim_{z_1 \to \infty} F_2(m, z_1, z_2)$$
$$- \lim_{z_2 \to \infty} F_2(m, z_1, z_2) + \lim_{\substack{z_1 \to \infty \\ z_2 \to \infty}} F_2(m, z_1, z_2)\} \tag{A.3.5}$$

Proof: By definition,

$$\mathcal{Z}\left[f_2(\langle k_1+m+1\rangle T, \langle k_2+m+1\rangle T)\right] = \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f_2(\langle k_1+m+1\rangle T, \langle k_2+m+1\rangle T)\, z_1^{-k_1} z_2^{-k_2}$$

$$= z_1 z_2 \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} f_2(<k_1+m+1>T, <k_2+m+1>T)\; z_1^{-(k_1+1)}\; z_2^{-(k_2+1)}$$

Letting $(k_1+1) = i_1$ and $(k_2+1) = i_2$ and using the initial value theorem gives eqn.(A.3.5), as required.

A.3.1.3 Complex Translation Theorem

If $F_n(m,z_1,z_2,\ldots,z_n)$ is the M.D.M.Z.T of $f_n(t_1,t_2,\ldots,t_n)$, then

$$Z\left[\left(\prod_{r=1}^{n} e^{-a_r(k_r+m)T}\right) f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)\right]$$
$$= \left(\prod_{r=1}^{n} e^{-a_r mT}\right) F_n(m, z_1 e^{a_1T}, z_2 e^{a_2T}, \ldots, z_n e^{a_nT})$$

Proof: By definition,

$$Z\left[\left(\prod_{r=1}^{n} e^{-a_r(k_r+m)T}\right) f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)\right]$$
$$= \left(\prod_{r=1}^{n} e^{-a_r mT}\right) \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T) \times \prod_{r=1}^{n} (z_r\, e^{a_rT})^{-k_r}$$
$$= \left(\prod_{r=1}^{n} e^{-a_r mT}\right) F_n(m, z_1 e^{a_1T}, z_2 e^{a_2T}, \ldots, z_n e^{a_nT}) \tag{A.3.6}$$

Consequently, the M.D.M.Z.T of $\left(\prod_{r=1}^{n} a_r^{i_r t_r}\right) f_n(t_1,t_2,\ldots,t_n)$ is given by

$$Z\left[\left(\prod_{r=1}^{n} a_r^{i_r(k_r+m)T}\right) f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)\right]$$
$$= \left(\prod_{r=1}^{n} a^{i_r mT}\right) F_n\left(m, \frac{z_1}{a^{i_1T}}, \frac{z_2}{a^{i_2T}}, \ldots, \frac{z_n}{a^{i_nT}}\right) \tag{A.3.7}$$

A.3.1.4 Final Value Theorem

If $F_n(m,z_1,z_2,\ldots,z_n)$ is the M.D.M.Z.T of $f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)$ and if the final value exists, then the final value of $f_1(kT)$ is given by

$$f_1(\infty) = \lim_{\substack{k_r \to \infty \\ 1 \le r \le n}} f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)$$
$$= \lim_{\substack{z_r \to 1 \\ 1 \le r \le n}} \left(\prod_{r=1}^{n} \frac{z_r - 1}{z_r}\right) F_n(m, z_1, z_2, \ldots, z_n) \tag{A.3.8}$$

When there is a ripple or hidden oscillation[88], the final value may be a function of m. The final value between sampling instants is then determined for $0 \leq m < 1$. If there is no ripple, the final value is found from the above equation by letting m=0.

Proof: The theorem is first proved for the two dimensional case to avoid complexity and is then easily extended to n dimensional case. To prove the final value theorem, consider the following expression

$$\lim_{\substack{i_1\to\infty \\ i_2\to\infty}} \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \{f_2(<k_1+m>T,<k_2+m>T)- f_2(<k_1+m>T,<k_2+m-1>T) - f_2(<k_1+m-1>T,<k_2+m>T)+ f_2(<k_1+m-1>T,<k_2+m-1>T)\} \prod_{r=1}^{2} z_r^{-k_r}$$

$$= \mathcal{Z}\left[f_2(<k_1+m>T,<k_2+m>T)- f_2(<k_1+m>T,<k_2+m-1>T)- f_2(<k_1+m-1>T,<k_2+m>T) + f_2(<k_1+m-1>T,<k_2+m-1>T)\right] \quad (A.3.9)$$

Using the backward shifting theorem and rearranging the terms, the right hand side of the above expression becomes

$$= \{ \prod_{r=1}^{2} (\frac{z_r-1}{z_r}) \} \; F_2(m,z_1,z_2)$$

Taking the limit as $z_1\to 1$ and $z_2\to 1$ on either side of eqn.(A.3.9) and interchanging limits on i_1, i_2 and z_1, z_2 yields

$$\lim_{\substack{i_1\to\infty \\ i_2\to\infty}} \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \{ f_2(<k_1+m>T,<k_2+m>T)- f_2(<k_1+m>T,<k_2+m-1>T) - f_2(<k_1+m-1>T,<k_2+m>T)+ f_2(<k_1+m-1>T,<k_2+m-1>T)\}$$

$$= \lim_{\substack{z_1\to 1 \\ z_2\to 1}} (\prod_{r=1}^{2} \frac{z_r-1}{z_r}) F_2(m,z_1,z_2) \quad (A.3.10)$$

Extending this to the n-dimensional case, gives eqn.(A.3.8), as required.

A.3.1.5 Summation Theorem

If $F_n(m,z_1,z_2,\dots,z_n)$ is the M.D.M.Z.T of $f_n(t_1,t_2,\dots,t_n)$, then

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \dots \sum_{k_n=0}^{\infty} f_n(<k_1+m>T,<k_2+m>T,\dots,<k_n+m>T)$$

$$= \lim_{\substack{z_r\to 1 \\ 1\leq r\leq n}} F_n(m,z_1,z_2,\dots,z_n) \quad (A.3.11)$$

Proof: Taking the limit, on either side of eqn.(A.3.1), as $z_1\to 1$ and taking the limit inside the summation on the left hand side, yields

$$\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T) \prod_{r=2}^{n} z_r^{-k_r}$$

$$= \lim_{z_1 \to 1} F_n(m, z_1, z_2, \ldots, z_n)$$

Similarly, taking the limit as $z_2 \to 1$, $z_3 \to 1, \ldots,$ and $z_n \to 1$, yields eqn. (A.3.11), as required.

Corollary: Finite Summation Theorem:

If $F_n(m, z_1, z_2, \ldots, z_n)$ is the M.D.M.Z.T of $f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)$ and if the function $g_n(<i_1+m>T, <i_2+m>T, \ldots, <i_n+m>T)$ given by

$g_n(<i_1+m>T, <i_2+m>T, \ldots, <i_n+m>T)$

$$= \sum_{k_1=0}^{i_1} \sum_{k_2=0}^{i_2} \cdots \sum_{k_n=0}^{i_n} f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T) \ , \qquad (A.3.12)$$

has a M.D.M.Z.T $G_n(m, z_1, z_2, \ldots, z_n)$, then

$$G_n(m, z_1, z_2, \ldots, z_n) = \left(\prod_{r=1}^{n} \frac{z_r}{(z_r - 1)} \right) F_n(m, z_1, z_2, \ldots, z_n) \qquad (A.3.13)$$

A.3.1.6 Complex Convolution Theorem

If $F_n(m, z_1, z_2, \ldots, z_n)$ and $G_n(m, z_1, z_2, \ldots, z_n)$ are the M.D.M.Z.T's of $f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)$ and $g_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)$, respectively, then

$$Z\left[f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T) \ g_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)\right]$$

$$= \frac{1}{(2\pi j)^n} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_n} F_n(m, w_1, w_2, \ldots, w_n) \ G_n\left(m, \frac{z_1}{w_1}, \frac{z_2}{w_2}, \ldots, \frac{z_n}{w_n}\right) \prod_{r=1}^{n} \frac{dw_r}{w_r} \qquad (A.3.14)$$

where C_r separates the poles of $F_n(m, w_1, w_2, \ldots, w_n)$ from those of $G_n(m, z_1/w_1, z_2/w_2, \ldots, z_n/w_n)$.

Proof: By definition,

$$Z\left[f_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T) \ g_n(<k_1+m>T, <k_2+m>T, \ldots, <k_n+m>T)\right]$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f_n(<k_1+m>T, \ldots, <k_n+m>T) g_n(<k_1+m>T, \ldots, <k_n+m>T) \times \prod_{r=1}^{n} z_r^{-k_r} \qquad (A.3.15)$$

But, by definition of M.D.I.M.Z.T.

$$f_n(<k_1+m>T,<k_2+m>T,\ldots,<k_n+m>T) = \frac{1}{(2\pi j)^n} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_n} F_n(m,w_1,w_2,\ldots,w_n) \times \prod_{r=1}^{n} w_r^{k_r-1} \, dw_r \qquad (A.3.16)$$

Substituting this in the above equation and changing the order of summation and integration and rearranging, yields

$$= \frac{1}{(2\pi j)^n} \oint_{C_1} \oint_{C_2} \cdots \oint_{C_n} F_n(m,w_1,w_2,\ldots,w_n)$$

$$\times \left[\sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} g_n(<k_1+m>T,\ldots,<k_n+m>T) \prod_{r=1}^{n} \left(\frac{z_r}{w_r}\right)^{-k_r} \right] \prod_{r=1}^{n} \frac{dw_r}{w_r}$$

Recognising that the term in the square bracket is $G_n(m,z_1/w_1,z_2/w_2,\ldots,z_n/w_n)$, yields eqn.(A.3.14), as required.

A.3.1.7 Partial Differentiation Theorem

If $F_n(m,a_1,z_1,a_2,z_2,\ldots,a_n,z_n)$ is the M.D.M.Z.T of $f_n(a_1,<k_1+m>T,a_2,<k_2+m>T,\ldots,a_n,<k_n+m>T)$, then

$$\mathcal{Z}\left[\left(\prod_{r=1}^{n} \frac{\partial}{\partial a_r}\right) f_n(a_1,<k_1+m>T,a_2,<k_2+m>T,\ldots,a_n,<k_n+m>T)\right]$$

$$= \left(\prod_{r=1}^{n} \frac{\partial}{\partial a_r}\right) F_n(m,a_1,z_1,a_2,z_2,\ldots,a_n,z_n) \qquad (A.3.17)$$

where a_r, $r=1,2,\ldots,n$, are independent variables or constants.

Proof: By definition,

$$\mathcal{Z}\left[\left(\prod_{r=1}^{n} \frac{\partial}{\partial a_r}\right) f_n(a_1,<k_1+m>T,a_2,<k_2+m>T,\ldots,a_n,<k_n+m>T)\right]$$

$$= \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} \left(\prod_{r=1}^{n} \frac{\partial}{\partial a_r}\right) f_n(a_1,<k_1+m>T,\ldots,a_n,<k_n+m>T) \prod_{r=1}^{n} z_r^{-k_r}$$

Since the variables a_r, $r=1,2,\ldots,n$, are independent of the variables k_r, $r=1,2,\ldots,n$, the differentiation may be taken out of the summation, which gives

$$= \left(\prod_{r=1}^{n} \frac{\partial}{\partial a_r}\right)\left\{ \sum_{k_1=0}^{\infty} \sum_{k_2=0}^{\infty} \cdots \sum_{k_n=0}^{\infty} f_n(a_1,<k_1+m>T,\ldots,a_n,<k_n+m>T) \prod_{r=1}^{n} z_r^{-k_r} \right\}$$

$$= \left(\prod_{r=1}^{n} \frac{\partial}{\partial a_r}\right) F_n(m,a_1,z_1,a_2,z_2,\ldots,a_n,z_n) \qquad (A.3.18)$$

A.3.1.8 Properties of Multidimensional Modified Z Transforms

(a) If $F_n(m,z_1,z_2,\dots,z_n)$ is the M.D.M.Z.T of $f_n(t_1,t_2,\dots,t_n)$, then

$$\mathcal{Z}\left[(k_r+m)T\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\right]$$
$$=\{mT\ F_n(m,z_1,z_2,\dots,z_n) - T\ z_r \frac{\partial}{\partial z_r} F_n(m,z_1,z_2,\dots,z_n)\},\quad 1\le r\le n$$

Proof: By definition, $\mathcal{Z}\left[(k_r+m)T\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\right]$

$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}(k_r+m)T\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\prod_{p=1}^{n} z_p^{-k_p}$$
$$= mT\ F_n(m,z_1,z_2,\dots,z_n)+\sum_{k_1=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}k_rT\ f_n(\langle k_1+m\rangle T,\dots,\langle k_n+m\rangle T)\prod_{p=1}^{n} z_p^{-k_p}$$
$$=\{mT\ F_n(m,z_1,z_2,\dots,z_n) - T\ z_r\frac{\partial}{\partial z_r}F_n(m,z_1,z_2,\dots,z_n)\},\quad 1\le r\le n \tag{A.3.19}$$

Consequently, the M.D.M.Z.T. of $(\prod_{r=1}^{n} t_r)\ f_n(t_1,t_2,\dots,t_n)$ is given by

$$\mathcal{Z}\left[(\prod_{r=1}^{n} k_rT)\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\right]$$
$$=(-1)^n\ (\prod_{r=1}^{n} T\ z_r\frac{\partial}{\partial z_r})\ F_n(m,z_1,z_2,\dots,z_n) \tag{A.3.20}$$

(b) If $F_n(m,z_1,z_2,\dots,z_n)$ and $G_n(m,z_1,z_2,\dots,z_n)$ are the M.D.M.Z.T's of $f_n(t_1,t_2,\dots,t_n)$ and $g_n(t_1,t_2,\dots,t_n)$, respectively, then the linearity relationship states that

$$\mathcal{Z}\left[a_1\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\pm a_2\ g_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\right]$$
$$= a_1\ F_n(m,z_1,z_2,\dots,z_n)\pm a_2\ G_n(m,z_1,z_2,\dots,z_n), \tag{A.3.21}$$

where a_1 and a_2 are constants.

(c) If $F_n(m,z_1,z_2,\dots,z_n)$ is the M.D.M.Z.T of $f_n(t_1,t_2,\dots,t_n)$, then

$$\mathcal{Z}\left[(\prod_{r=1}^{n}\frac{1}{k_rT})\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\right]$$
$$=(\prod_{r=1}^{n}\frac{-1}{Tz_r})\int\int\cdots\int F_n(m,z_1,z_2,\dots,z_n)\prod_{r=1}^{n} dz_r \tag{A.3.22}$$

Proof: By definition,

$$\mathcal{Z}\left[(\prod_{r=1}^{n}\frac{1}{k_rT})\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\right]$$
$$=\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty}(\prod_{r=1}^{n}\frac{1}{k_rT})\ f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\prod_{r=1}^{n} z_r^{-k_r}$$
$$=(\prod_{r=1}^{n}\frac{-1}{Tz_r})\int\int\cdots\int\sum_{k_1=0}^{\infty}\sum_{k_2=0}^{\infty}\cdots\sum_{k_n=0}^{\infty} f_n(\langle k_1+m\rangle T,\langle k_2+m\rangle T,\dots,\langle k_n+m\rangle T)\times\prod_{r=1}^{n} z_r^{-k_r}\,dz_r$$
$$=(\prod_{r=1}^{n}\frac{-1}{Tz_r})\int\int\cdots\int F_n(m,z_1,z_2,\dots,z_n)\prod_{r=1}^{n} dz_r.$$

Appendix A.3.2. Tables of Multidimensional Z and Modified Z and their Associated Z Transforms.

No.	M.D.L.T	M.D.Z.T	A.Z.T	M.D.M.Z.T	A.M.Z.T
1.	$\frac{k}{s_1 s_2}$	$\frac{k\, z_1 z_2}{(z_1-1)(z_2-1)}$	$\frac{k\, z}{(z-1)}$	$\frac{k\, z_1 z_2}{(z_1-1)(z_2-1)}$	$\frac{k\, z}{(z-1)}$
2.	$\frac{k}{(s_1+\alpha)(s_2+\beta)}$	$\frac{k\, z_1 z_2}{(z_1-e^{-\alpha T})(z_2-e^{-\beta T})}$	$\frac{k\, z}{(z-e^{-(\alpha+\beta)T})}$	$\frac{k\, e^{-(\alpha+\beta)mT}\, z_1 z_2}{(z_1-e^{-\alpha T})(z_2-e^{-\beta T})}$	$\frac{kz\, e^{-(\alpha+\beta)mT}}{(z-e^{-(\alpha+\beta)T})}$
3.	$\frac{k}{s_1 s_2(s_1+s_2)}$	$\frac{kTz_1 z_2}{(z_1 z_2-1)(z_1-1)(z_2-1)}$	$\frac{kTz}{(z-1)^2}$	$\frac{kTz_1 z_2\{m(z_1 z_2-1)+1\}}{(z_1 z_2-1)(z_1-1)(z_2-1)}$	$\frac{kTz\{mz+(1-m)\}}{(z-1)^2}$
4.	$\frac{k}{s_1 s_2(s_1+\alpha) \times (s_2+\beta)}$	$\frac{kz_1 z_2(1-e^{-\alpha T})(1-e^{-\beta T})}{\alpha\beta(z_1-1)(z_2-1)(z_1-e^{-\alpha T}) \times (z_2-e^{-\beta T})}$	$\frac{kz(1-e^{-\alpha T})(1-e^{-\beta T})}{\alpha\beta(z-e^{-(\alpha+\beta)T}) \times (z-1)}$	$\frac{\left[kz_1 z_2\{z_2(1-e^{-m\beta T})+(e^{-m\beta T}-e^{-\beta T})\}\{z_1(1-e^{-m\alpha T})+(e^{-m\alpha T}-e^{-\alpha T})\}\right]}{\alpha\beta(z_1-1)(z_2-1)(z_1-e^{-\alpha T}) \times (z_2-e^{-\beta T})}$	$\frac{kz\, e^{-(\alpha+\beta)mT}(1-e^{-\alpha T}) \times (1-e^{-\beta T})}{\alpha\beta(z-e^{-(\alpha+\beta)T}) \times (z-1)}$
5.	$\frac{k}{(s_1+s_2+\alpha)}$	$\frac{kz_1 z_2}{(z_1 z_2-e^{-\alpha T})}$	$\frac{kz}{(z-e^{-\alpha T})}$	$\frac{kz_1 z_2 e^{-m\alpha T}}{(z_1 z_2-e^{-\alpha T})}$	$\frac{kz\, e^{-m\alpha T}}{(z-e^{-\alpha T})}$
6.	$\frac{k}{s_1 s_2(s_1+s_2+a)}$	$\frac{kz_1 z_2(1-e^{-aT})}{a(z_1 z_2-e^{-aT})(z_1-1) \times (z_2-1)}$	$\frac{kz(1-e^{-aT})}{a(z-1)(z-e^{-aT})}$	$\frac{kz_1 z_2\{z_1 z_2(1-e^{-amT}) + (e^{-amT}-e^{-aT})\}}{a(z_1 z_2-e^{-aT})(z_1-1)(z_2-1)}$	$\frac{kz\{z(1-e^{-amT}) + (e^{-amT}-e^{-aT})\}}{a(z-1)(z-e^{-aT})}$

7.	$\dfrac{k}{(s_1+s_2+a)(s_1+b)\times(s_2+c)}$	$\dfrac{kz_1z_2\{e^{-(b+c)T}-e^{-aT}\}}{(a-b-c)(z_1z_2-e^{-aT})\times(z_1-e^{-bT})(z_2-e^{-cT})}$	$\dfrac{kz(e^{-(b+c)T}-e^{-aT})}{(a-b-c)(z-e^{-aT})\times(z-e^{-(b+c)T})}$	$\dfrac{kz_1z_2\{e^{-(b+c)mT}(z_1z_2-e^{-aT})-e^{-amT}(z_1z_2-e^{-(b+c)T})\}}{(a-b-c)(z_1z_2-e^{-aT})(z_1-e^{-bT})\times(z_2-e^{-cT})}$	$\dfrac{kz\{e^{-(b+c)mT}(z-e^{-aT})-e^{-amT}(z-e^{-(b+c)T})\}}{(a-b-c)(z-e^{-aT})\times(z-e^{-(b+c)T})}$
8.	$\dfrac{k}{s_1^2s_2^2}$	$\dfrac{kT^2z_1z_2}{(z_1-1)^2(z_2-1)^2}$	$\dfrac{kT^2z(z+1)}{(z-1)^3}$	$\dfrac{kT^2z_1z_2\{m(z_1-1)+1\}\times\{m(z_2-1)+1\}}{(z_1-1)^2(z_2-1)^2}$	$\dfrac{kT^2z\left[\{mz+(1-m)\}^2+z\right]}{(z-1)^3}$
9.	$\dfrac{k}{(s_1+a)(s_2^2+b^2)}$	$\dfrac{kz_1z_2\sin bT}{b(z_1-e^{-aT})(z_2-e^{-jbT})\times(z_2-e^{jbT})}$	$\dfrac{kze^{-aT}\sin bT}{b(z-e^{-(a-jb)T})\times(z-e^{-(a+jb)T})}$	$\dfrac{kz_1z_2e^{-amT}\left[z_2\sin mbT-\sin\{bT(m-1)\}\right]}{b(z_1-e^{-aT})(z_2-e^{-jbT})(z_2-e^{jbT})}$	$\dfrac{kze^{-amT}\{z\sin mbT-e^{-aT}\sin bT(m-1)\}}{b(z-e^{-(a-jb)T})\times(z-e^{-(a+jb)T})}$
10.	$\dfrac{ks_2}{(s_1+a)(s_2^2+b^2)}$	$\dfrac{kz_1z_2\{z_2-\cos bT\}}{(z_1-e^{-aT})(z_2-e^{-jbT})\times(z_2-e^{jbT})}$	$\dfrac{kz(z-e^{-aT}\cos bT)}{(z-e^{-(a-jb)T})\times(z-e^{-(a+jb)T})}$	$\dfrac{kz_1z_2e^{-amT}\{z_2\cos mbT-\cos[bT(m-1)]\}}{(z_1-e^{-aT})(z_2-e^{-jbT})(z_2-e^{jbT})}$	$\dfrac{kze^{-amT}\{z\cos mbT-e^{-aT}\cos bT(m-1)\}}{(z-e^{-(a-jb)T})\times(z-e^{-(a+jb)T})}$
11.	$\dfrac{k}{(s_1+a)(s_2+b)^2}$	$\dfrac{kTz_1z_2e^{-bT}}{(z_1-e^{-aT})(z_2-e^{-bT})^2}$	$\dfrac{kTze^{-(a+b)T}}{(z-e^{-(a+b)T})^2}$	$\dfrac{kTz_1z_2e^{-(a+b)mT}\{mz_2+e^{-bT}(1-m)\}}{(z_1-e^{-aT})(z_2-e^{-bT})^2}$	$\dfrac{kTze^{-(a+b)mT}\times\{mz+(1-m)e^{-(a+b)T}\}}{(z-e^{-(a+b)T})^2}$
12.	$\dfrac{k}{(s_1+s_2+s_3+a)}$	$\dfrac{kz_1z_2z_3}{(z_1z_2z_3-e^{-aT})}$	$\dfrac{kz}{(z-e^{-aT})}$	$\dfrac{kz_1z_2z_3e^{-amT}}{(z_1z_2z_3-e^{-aT})}$	$\dfrac{kze^{-amT}}{(z-e^{-aT})}$

13.	$\frac{k}{(s_1+a)(s_2+s_3+a)\times(s_1+s_2+s_3+a)}$	$\frac{ke^{-aT}(1-e^{-aT})z_1z_2z_3}{a(z_1-e^{-aT})(z_2z_3-e^{-aT})\times(z_1z_2z_3-e^{-aT})}$	$\frac{kze^{-aT}(1-e^{-aT})}{a(z-e^{-2aT})\times(z-e^{-aT})}$	$\frac{ke^{-amT}z_1z_2z_3\{z_1z_2z_3(1-e^{-amT})+e^{-aT}(e^{-amT}-e^{-aT})\}}{a(z_1-e^{-aT})(z_2z_3-e^{-aT})\times(z_1z_2z_3-e^{-aT})}$	$\frac{kze^{-amT}\{z(1-e^{-amT})+e^{-aT}(e^{-amT}-e^{-aT})\}}{a(z-e^{-aT})(z-e^{-2aT})}$
14.	$\frac{k}{(s_1+s_2+s_3+b)(s_1+a)\times(s_2+a)(s_3+a)}$	$\frac{kz_1z_2z_3(e^{-3aT}-e^{-bT})}{(b-3a)(z_1z_2z_3-e^{-bT})\times(z_1-e^{-aT})(z_2-e^{-aT})(z_3-e^{-aT})}$	$\frac{kz(e^{-3aT}-e^{-bT})}{(b-3a)(z-e^{-3aT})\times(z-e^{-bT})}$	$\frac{kz_1z_2z_3\{e^{-3amT}(z_1z_2z_3-e^{-bT})-e^{-bmT}(z_1z_2z_3-e^{-3aT})\}}{(b-3a)(z_1z_2z_3-e^{-bT})(z_1-e^{-aT})(z_2-e^{-aT})(z_3-e^{-aT})}$	$\frac{kz\{e^{-3amT}(z-e^{-bT})-e^{-bmT}(z-e^{-3aT})\}}{(b-3a)(z-e^{-3aT})(z-e^{-bT})}$
15.	$\frac{k}{(s_1+s_2+s_3+a)s_1s_2s_3}$	$\frac{kz_1z_2z_3(1-e^{-aT})}{a(z_1z_2z_3-e^{-aT})(z_1-1)\times(z_2-1)(z_3-1)}$	$\frac{kz(1-e^{-aT})}{a(z-e^{-aT})(z-1)}$	$\frac{kz_1z_2z_3\{z_1z_2z_3(1-e^{-amT})+(e^{-amT}-e^{-aT})\}}{a(z_1z_2z_3-e^{-aT})(z_1-1)(z_2-1)\times(z_3-1)}$	$\frac{kz\{(z-e^{-aT})-e^{-amT}(z-1)\}}{a(z-e^{-aT})(z-1)}$
16.	$\frac{k}{(s_1+s_2+s_3+a)(s_1+a)\times(s_2+s_3+a)}$	$\frac{k\,z_1z_2z_3e^{-aT}(1-e^{-aT})}{a(z_1z_2z_3-e^{-aT})(z_1-e^{-aT})\times(z_2z_3-e^{-aT})}$	$\frac{kze^{-aT}(1-e^{-aT})}{a(z-e^{-2aT})\times(z-e^{-aT})}$	$\frac{kz_1z_2z_3e^{-amT}\{z_1z_2z_3(1-e^{-amT})+e^{-aT}(e^{-amT}-e^{-aT})\}}{a(z_1z_2z_3-e^{-aT})(z_2z_3-e^{-aT})\times(z_1-e^{-aT})}$	$\frac{kze^{-amT}\{z(1-e^{-amT})+e^{-aT}(e^{-amT}-e^{-aT})\}}{a(z-e^{-2aT})\times(z-e^{-aT})}$

17.	$\dfrac{k}{(s_2+s_3+a)(s_1+a) \times s_1 s_2 s_3}$	$\dfrac{kz_1z_2z_3(1-e^{-aT})^2}{a^2(z_2z_3-e^{-aT})(z_1-e^{-aT}) \times (z_1-1)(z_2-1)(z_3-1)}$	$\dfrac{kz(z+e^{-aT})(1-e^{-aT})^2}{a^2(z-e^{-2aT})(z-1) \times (z-e^{-aT})}$	$\dfrac{kz_1z_2z_3\{z_1(1-e^{-amT})+(e^{-amT}-e^{-aT})\}\{z_2z_3(1-e^{-amT})+(e^{-amT}-e^{-aT})\}}{a^2(z_2z_3-e^{-aT})(z_1-e^{-aT})(z_1-1) \times (z_2-1)(z_3-1)}$	$\dfrac{kz}{a^2}\left[\dfrac{e^{-2amT}}{(z-e^{-2aT})} + \dfrac{1}{(z-1)} - \dfrac{2e^{-amT}}{(z-e^{-aT})}\right]$
18.	$\dfrac{k}{(s_2+s_3+a)(s_1+a)}$	$\dfrac{kz_1z_2z_3}{(z_2z_3-e^{-aT})(z_1-e^{-aT})}$	$\dfrac{kz}{(z-e^{-2aT})}$	$\dfrac{kz_1z_2z_3e^{-2amT}}{(z_1-e^{-aT})(z_2z_3-e^{-aT})}$	$\dfrac{kze^{-2amT}}{(z-e^{-2aT})}$
19.	$\dfrac{k}{(s_1+a)(s_2+a) \times (s_3+a)}$	$\dfrac{kz_1z_2z_3}{(z_1-e^{-aT})(z_2-e^{-aT}) \times (z_3-e^{-aT})}$	$\dfrac{kz}{(z-e^{-3aT})}$	$\dfrac{kz_1z_2z_3e^{-3amT}}{(z_1-e^{-aT})(z_2-e^{-aT})(z_3-e^{-aT})}$	$\dfrac{kze^{-3amT}}{(z-e^{-3aT})}$
20.	$\dfrac{k}{s_1^2s_2^2s_3^2}$	$\dfrac{kT^3z_1z_2z_3}{(z_1-1)^2(z_2-1)^2(z_3-1)^2}$	$\dfrac{kT^3z(z^2+4z+1)}{(z-1)^4}$	$\dfrac{kT^3z_1z_2z_3\{mz_1+(1-m)\}\{mz_2+(1-m)\}\{mz_3+(1-m)\}}{(z_1-1)^2(z_2-1)^2(z_3-1)^2}$	$\dfrac{kT^3z\left[(z-1)^2\{m^3 \times (z-1)+3m^2+3m+1\} + 6\{(m+1)(z-1)+1\}\right]}{(z-1)^4}$
21.	$\dfrac{k}{(s_3^2+c^2)(s_1+a) \times (s_2+b)}$	$\dfrac{k\ z_1z_2z_3\ \sin cT}{c(z_1-e^{-aT})(z_2-e^{-bT}) \times (z_3-e^{-jcT})(z_3-e^{+jcT})}$	$\dfrac{kze^{-(a+b)T}\sin cT}{c(z-e^{-(a+b+jc)T}) \times (z-e^{-(a+b-jc)T})}$	$\dfrac{kz_1z_2z_3e^{-(a+b)mT}\{z_3\sin mcT - \sin(m-1)cT\}}{c(z_1-e^{-aT})(z_2-e^{-bT})(z_3-e^{jcT}) \times (z_3-e^{-jcT})}$	$\dfrac{kz\ e^{-m(a+b)T}\{z\ \sin mcT-e^{-(a+b)T}\sin(m-1)cT\}}{c(z-e^{-(a+b+jc)T}) \times (z-e^{-(a+b-jc)T})}$

22.	$\frac{ks_3}{(s_1+a)(s_2+b)\times(s_3^2+c^2)}$	$\frac{kz_1z_2z_3\{z_3-\cos cT\}}{(z_1-e^{-aT})(z_2-e^{-bT})\times(z_3-e^{-jcT})(z_3-e^{jcT})}$	$\frac{kz\{z-e^{-(a+b)T}\cos cT\}}{(z-e^{-(a+b+jc)T})\times(z-e^{-(a+b-jc)T})}$	$\frac{kz_1z_2z_3e^{-(a+b)mT}\{z_3\cos mcT-\cos(m-1)cT\}}{(z_1-e^{-aT})(z_2-e^{-bT})\times(z_3-e^{-jcT})(z_3-e^{jcT})}$	$\frac{kze^{-(a+b)mT}\{z\cos mcT-e^{-(a+b)T}\cos(m-1)cT\}}{(z-e^{-(a+b+jc)T})\times(z-e^{-(a+b-jc)T})}$
23.	$\frac{k}{s_1s_2s_3(s_1+a)\times(s_2+a)(s_3+a)}$	$\frac{kz_1z_2z_3(1-e^{-aT})^3}{a^3(z_1-1)(z_1-e^{-aT})(z_2-1)\times(z_2-e^{-aT})(z_3-1)(z_3-e^{-aT})}$	$\frac{(1-e^{-aT})^3\{z(z+2e^{-aT})+e^{-2aT}(2z+e^{-aT})\}kz}{a^3(z-1)(z-e^{-aT})\times(z-e^{-2aT})(z-e^{-3aT})}$	$\frac{kz_1z_2z_3\prod_{r=1}^{3}\{(z_r-e^{-aT})-e^{-amT}(z_r-1)\}}{a^3(z_1-1)(z_1-e^{-aT})(z_2-1)\times(z_2-e^{-aT})(z_3-1)(z_3-e^{-aT})}$	$\frac{kz}{a^3}\left[\frac{1}{z-1}-\frac{3e^{-amT}}{z-e^{-aT}}+\frac{3e^{-2amT}}{z-e^{-2aT}}-\frac{e^{-3amT}}{z-e^{-3aT}}\right]$
24.	$\frac{k}{(s_1+a)(s_2^2+b^2)\times(s_3^2+c^2)}$	$\frac{kz_1z_2z_3\sin bT\sin cT}{bc(z_1-e^{-aT})\times(z_2-e^{-jbT})(z_2-e^{jbT})\times(z_3-e^{-jcT})(z_3-e^{jcT})}$	$\frac{kz}{2bc}\left[\frac{z-e^{-aT}\cos(b-c)T}{(z-e^{-\{a+j(b-c)\}T})\times(z-e^{-\{a-j(b-c)\}T})}-\frac{z-e^{-aT}\cos(b+c)T}{(z-e^{-\{a+j(b+c)\}T})\times(z-e^{-\{a-j(b+c)\}T})}\right]$	$\frac{kz_1z_2z_3e^{-amT}\{z_2\sin mbT-\sin(m-1)bT\}\{z_3\sin mcT-\sin(m-1)cT\}}{bc(z_1-e^{-aT})(z_2-e^{-jbT})\times(z_2-e^{jbT})(z_3-e^{-jcT})\times(z_3-e^{jcT})}$	$\left[\frac{kze^{-amT}\{z\cos(b-c)mT-e^{-aT}\cos(m-1)(b-c)T\}}{2(z-e^{-\{a+j(b-c)\}T})b\times(z-e^{-\{a-j(b-c)\}T})c}-\frac{kz\,e^{-amT}\{z\cos(b+c)mT-e^{-aT}\times\cos(m-1)(b+c)T\}}{2(z-e^{-\{a+j(b+c)\}T})b\times(z-e^{-\{a-j(b+c)\}T})c}\right]$

APPENDIX A.4

ANALYSIS OF NONLINEAR FEEDBACK SYSTEMS WITH VARIOUS LOCATIONS OF SAMPLERS

The multidimensional z transform kernels characterising the three types of nonlinear feedback systems, with various locations of samplers, are derived here. Since, the procedure of obtaining the z transform kernels depends on the location as well as on the presence or the absence, of the sampler, each feedback system is treated separately.

A.4.1 Feedback system of type 1

The type 1 feedback system is shown in Fig.A.4.1(a), in which J and K are nonlinear subsystems and the input to J and the output of K are sampled. The equivalent system is shown in Fig.A.4.1(b). The system equations[95] may be written in operator form as

$$M^* = I - N^* \circledast M^* \quad , \tag{A.4.1a}$$

$$L^* = J^* \circledast M^* \tag{A.4.1b}$$

where I is an Identity Operator and $N^* = (K \circledast J)^* \neq K^* \circledast J^*$, since there is no sampler between K and J. The solution of eqns.(A.4.1) gives the z transform kernels of L which represents the system shown in Fig.A.4.1(a).

The cascade combination of continuous systems K and J in Laplace transform domain gives the kernels of N as

$$N_1(s) = K_1(s)J_1(s) \quad , \quad N_2(s_1,s_2) = K_1(s_1+s_2)J_2(s_1,s_2)+K_2(s_1,s_2)\prod_{k=1}^{2} J_1(s_k)$$

$$N_3(s_1,s_2,s_3) = K_1(s_1+s_2+s_3)J_3(s_1,s_2,s_3)+2K_2(s_1,s_2+s_3)J_1(s_1)\,J_2(s_2,s_3)$$
$$+ K_3(s_1,s_2,s_3)\prod_{k=1}^{3} J_1(s_k) \tag{A.4.2}$$

The kernels of the equivalent system L are given by the nonlinear system operation of eqn.(A.4.1b),

$$L_1(z) = J_1(z)M_1(z) \quad , \quad L_2(z_1,z_2) = J_2(z_1,z_2)\prod_{k=1}^{2} M_1(z_k)+ J_1(z_1z_2)M_2(z_1,z_2)$$

$$L_3(z_1,z_2,z_3) = J_3(z_1,z_2,z_3)\prod_{k=1}^{3} M_1(z_k)+2J_2(z_1,z_2z_3)M_1(z_1)M_2(z_2,z_3)$$
$$+ J_1(z_1z_2z_3)M_3(z_1,z_2,z_3) \tag{A.4.3}$$

where the components of M, $M_1(z)$, $M_2(z_1,z_2)$ and $M_3(z_1,z_2,z_3)$ are obtained from the solution of eqn.(A.4.1a), which gives

$$M_1(z) = \frac{1}{\{1+N_1(z)\}} \qquad M_2(z_1,z_2) = \frac{-N_2(z_1,z_2)M_1(z_1)M_1(z_2)}{\{1+N_1(z_1z_2)\}}$$

$$M_3(z_1,z_2,z_3) = \frac{-\{N_3(z_1,z_2,z_3)\prod_{k=1}^{3} M_1(z_k) + 2N_2(z_1z_2,z_3)M_2(z_1,z_2)M_1(z_3)\}}{\{1+N_1(z_1z_2z_3)\}} \tag{A.4.4}$$

where $N_1(z)$, $N_2(z_1,z_2)$ and $N_3(z_1,z_2,z_3)$ may be obtained by applying the sequential process to $N_1(s)$, $N_2(s_1,s_2)$ and $N_3(s_1,s_2,s_3)$ of eqn.(A.4.2), respectively.

To obtain the response between sampled instants (i.e.; continuous-time output), M.D.M.Z.T is to be used. Since the subsystem J receives the input only at the sampling instants and since its output is to be modified to obtain the inter-sampled response, it is sufficient to consider the modified z transform of J only. Thus, the modified z transform kernels of L are given by

$$L_1(m,z) = J_1(m,z)M_1(z) ,$$

$$L_2(m,z_1,z_2) = J_2(m,z_1,z_2)\prod_{k=1}^{2} M_1(z_k) + J_1(m,z_1z_2)M_2(z_1,z_2)$$

$$L_3(m,z_1,z_2,z_3) = J_3(m,z_1,z_2,z_3)\prod_{k=1}^{3} M_1(z_k) + 2J_2(m,z_1,z_2z_3)M_1(z_1)M_2(z_2,z_3)$$
$$+ J_1(m,z_1z_2z_3)M_3(z_1,z_2,z_3) \tag{A.4.5}$$

where the components of M and N are given by eqns.(A.4.4) and (A.4.2), respectively.

A.4.2 Feedback system of type 2

The feedback system of type 2 is shown in Fig.A.4.2(a) in which the inputs and the outputs of J and K are sampled. The equivalent system is shown in Fig.A.4.2(b). Since there is a sampler between J and K, N is given by the nonlinear system operation

$$N = K \circledast J \quad , \tag{A.4.6}$$

which gives the components of N as

$$N_1(z) = K_1(z)J_1(z) \ , \quad N_2(z_1,z_2) = K_1(z_1z_2)J_2(z_1,z_2) + K_2(z_1,z_2)\prod_{k=1}^{2} J_1(z_k)$$

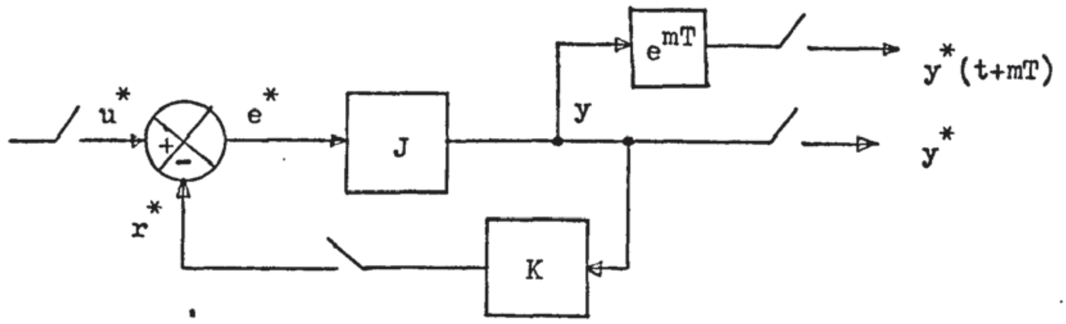


Fig.A.4.1(a) Feedback nonlinear sampled-data system without sampler between J and K.

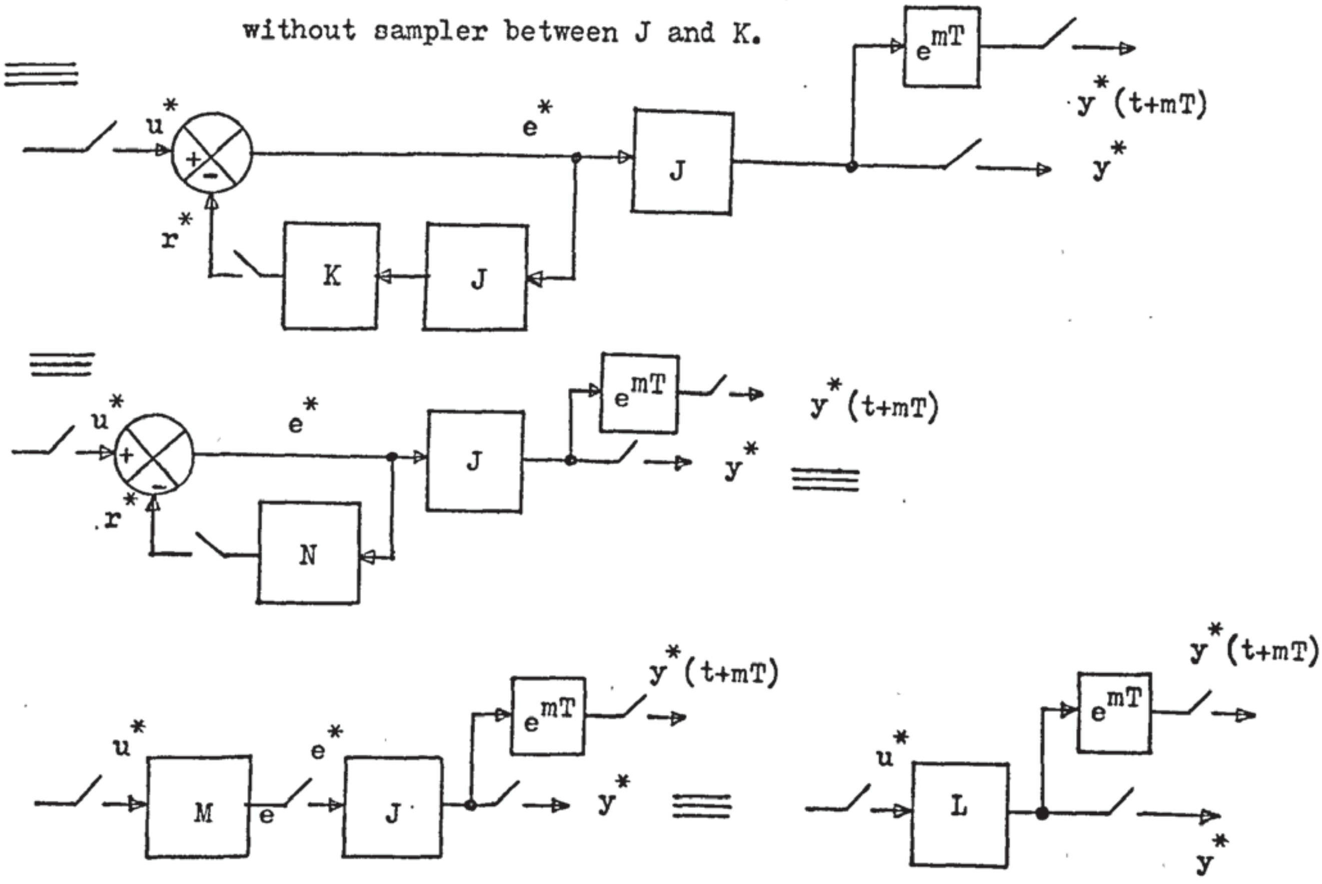


Fig.A.4.1(b) Equivalent system for the system shown in Fig.A.4.1(a).

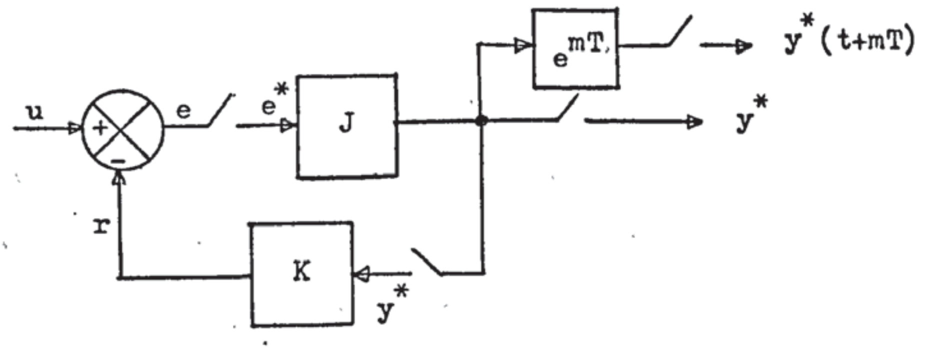


Fig.A.4.2(a) Feedback system with a sampler between J and K.

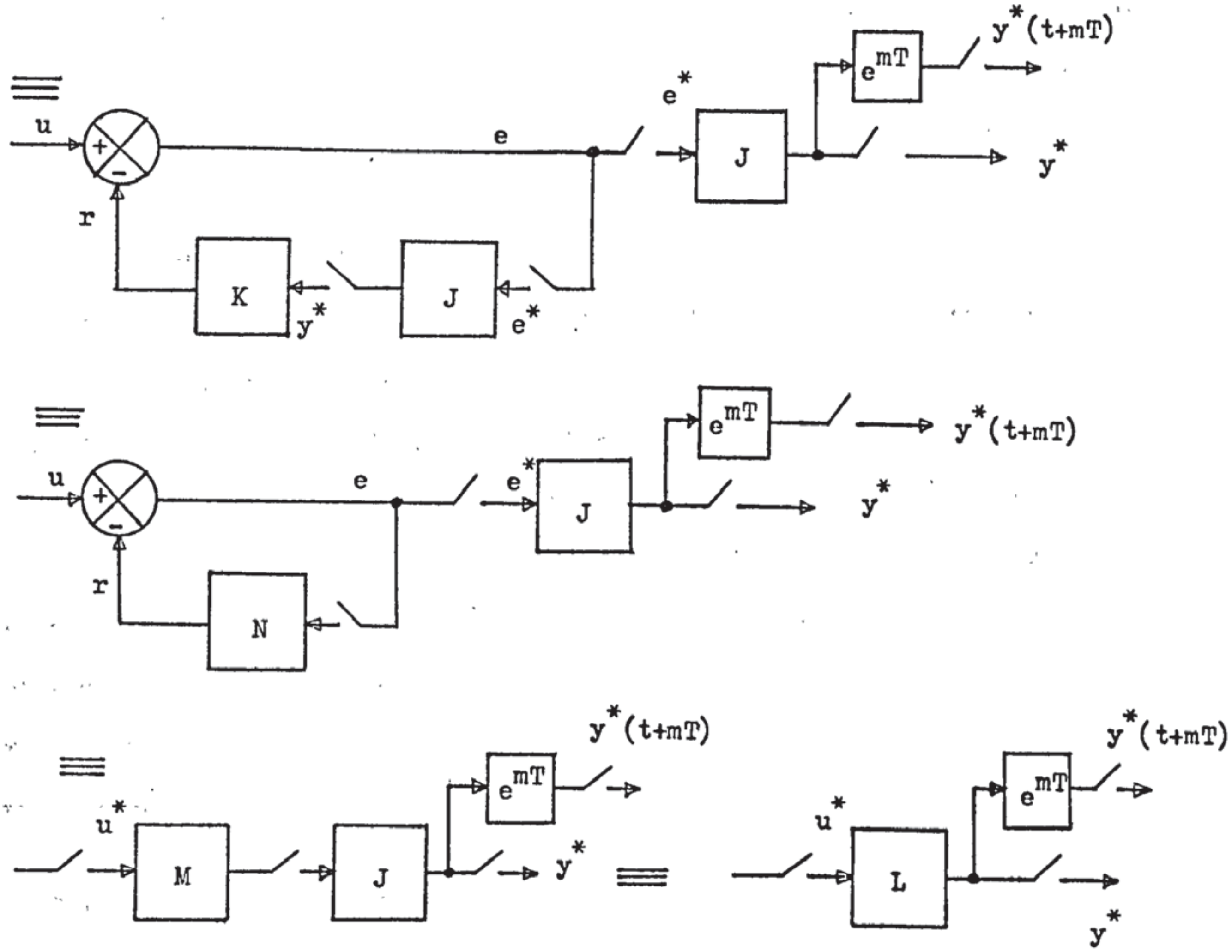


Fig.A.4.2(b) Equivalent system.

$$N_3(z_1,z_2,z_3) = K_1(z_1z_2z_3)J_3(z_1,z_2,z_3) + 2K_2(z_1,z_2z_3)J_1(z_1)J_2(z_2,z_3)$$
$$+ K_3(z_1,z_2,z_3)\prod_{k=1}^{3} J_1(z_k) \qquad \text{(A.4.7)}$$

Then, the z and the modified z transform kernels of the equivalent system L may be obtained from eqns.(A.4.3) to (A.4.5) and (A.4.7).

A.4.3 Feedback system of type 3

The type 3 feedback system is shown in Fig.A.4.3(a) in which, only the input and the output of the system are sampled. The equivalent system is shown in Fig.A.4.3(b). Since the system shown in Fig.A.4.3(a) is a continuous nonlinear feedback system with sampled input and output, the kernels of the equivalent open-loop system L, which is also a continuous nonlinear system, may be derived using the cascade relations of continuous-time systems. Then, the z transform kernels of the equivalent system may be obtained by applying the sequential process to the corresponding Laplace transform kernels. First, the M.D.L.T of kernels of L are derived.

The cascade combination of J and K is given by the cascade operation

$$N = K \otimes J \quad , \qquad \text{(A.4.8)}$$

whose components are given by eqns.(A.4.2). The components of M may be obtained by solving the equation

$$M = I - N \otimes M \qquad \text{(A.4.9)}$$

which gives $M_1(s) = \dfrac{1}{\{1 + N_1(s)\}}$,

$$M_2(s_1,s_2) = \frac{-N_2(s_1,s_2)M_1(s_1)M_1(s_2)}{\{1 + N_1(s_1+s_2)\}} \qquad \text{(A.4.10)}$$

$$M_3(s_1,s_2,s_3) = \frac{-\{N_3(s_1,s_2,s_3)\prod_{k=1}^{3} M_1(s_k) + 2N_2(s_1+s_2,s_3)M_2(s_1,s_2)M_1(s_3)\}}{\{1 + N_1(s_1+s_2+s_3)\}}$$

where $N_1(s)$, $N_2(s_1,s_2)$ and $N_3(s_1,s_2,s_3)$ are given by eqns.(A.4.2). Since M and J are not separated by a sampler, the equivalent system L is obtained by the cascade operation

$$L = J \otimes M \quad , \qquad \text{(A.4.11)}$$

which gives the components of L as

$$L_1(s) = J_1(s)M_1(s) \; , \quad L_2(s_1,s_2) = J_2(s_1,s_2)\prod_{k=1}^{2} M_1(s_k) + J_1(s_1+s_2)M_2(s_1,s_2)$$

and

$$L_3(s_1,s_2,s_3) = J_3(s_1,s_2,s_3)\prod_{k=1}^{3} M_1(s_k) + 2J_2(s_1,s_2+s_3)M_1(s_1)M_2(s_2,s_3)$$

$$+ J_1(s_1+s_2+s_3)M_3(s_1,s_2,s_3) \qquad (A.4.12)$$

Then, the z transform kernels $L_1(z)$, $L_2(z_1,z_2)$, $L_3(z_1,z_2,z_3)$ and the modified z transfrom kernels $L_1(m,z)$, $L_2(m,z_1,z_2)$ and $L_3(m,z_1,z_2,z_3)$ may be obtained by applying the sequential process of chapters 2 and 3 to $L_1(s)$, $L_2(s_1,s_2)$ and $L_3(s_1,s_2,s_3)$, respectively.

A.4.4 Feedback system with input not sampled

A feedback system, in which the input to subsystem J is not sampled, is shown in Fig.A.4.4, for which it is not possible to obtain an explicit input-output relationship in the z transform domain. However, the following system equations exist for this system.

$$e = u - K([J(e)]^*)$$
$$y^* = \{J(u - K(y^*))\}^* \qquad (A.4.13)$$

Even in the case where J and K are linear subsystems, there is no explicit input-output relation and the output z transform[83] is given by

$$Y_1(z) = \frac{N_1(z)}{\{1+ M_1(z)\}} \qquad (A.4.14)$$

where $N_1(z)$ and $M_1(z)$ are z transforms of $N_1(s)$ and $M_1(s)$, respectively and $N_1(s)$ and $M_1(s)$ are given by

$$M_1(s) = K_1(s)\ J_1(s)$$
$$N_1(s) = J_1(s)\ U_1(s) \qquad (A.4.15)$$

where $J_1(s)$ and $K_1(s)$ are linear transfer functions.

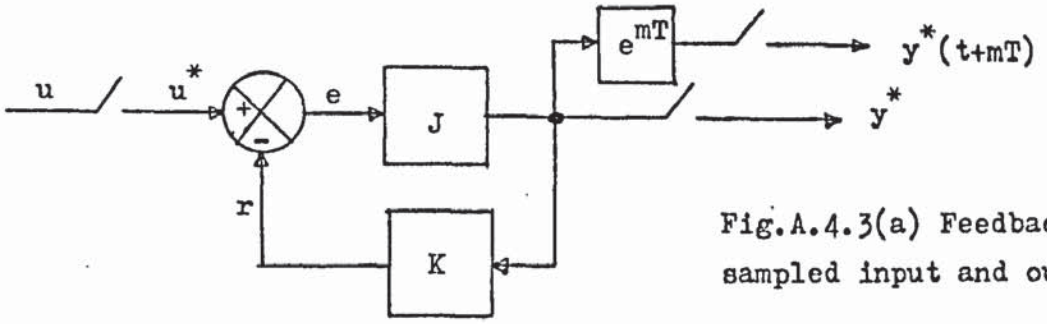


Fig.A.4.3(a) Feedback system with sampled input and output.

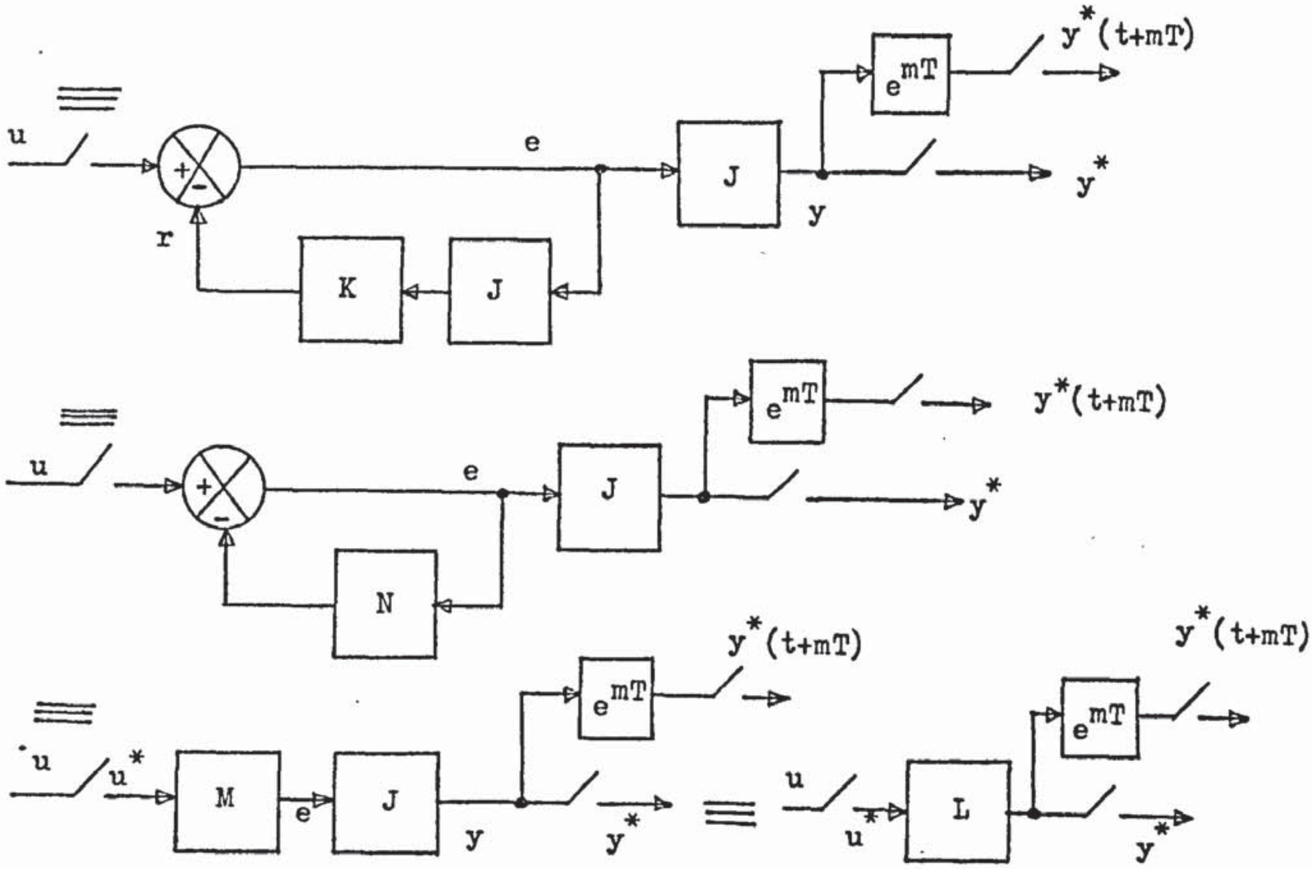


Fig.A.4.3(b) Equivalent system.

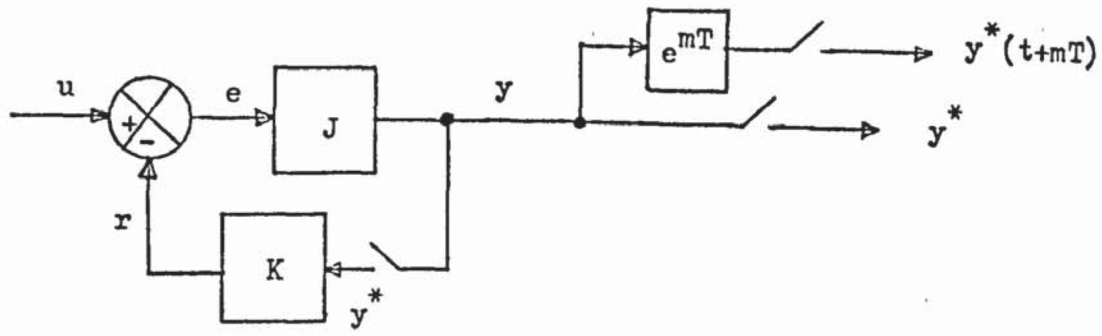


Fig.A.4.4 Feedback system with input not sampled.

APPENDIX A.5

Synthesis Of A Third-Order Kernel

No.	Test	Result	Conclusions
1.	Test $M_3(m,1,1,z)$, $M_3(m,1,z,1)$ and $M_3(m,z,1,1)$, for a common factor.	A common factor of the form $_5M_1(m,z)= ze^{-a_5mT}\{zf_5(m,b_5)- e^{-a_5T} f_6(m,c_5,b_5)\}$ exists in each of them.	$_5J_1(m,z)$ is a second-order linear system, and $f_5(m,b_5)=(\cos mb_5T+ c_5\sin mb_5T)$ $f_6(m,c_5,b_5)=\{\cos(1-m)b_5T - c_5\sin(1-m)b_5T\}$ (A.5.1)
		A common factor $_5M_1(m,z)= e^{-d_{51}mT}(d_{52}z \times e^{-d_{53}T} -d_{54}e^{-d_{55}T})$ exists.	$_5J_1(m,z)$ is a first or second-order linear system and d_{51} to d_{55} are determined by comparison.
		No such common factor exists.	$_5J_1(m,z)$ is either absent or a first or second-order system.
	Compute $M_3(m,1,z,z^{-1})$ and $M_3(m,z^2,z^{-1},z^{-1})$, if necessary.	$_5M_1(m,z)$ is a constant.	$_5J_1(m,z)$ is a first or second-order linear system.
2.	Divide $M_3(m,z_1,z_2,z_3)$ by $_5M_1(m,z_1z_2z_3)$, if it exists in step 1, to give $A_3(z_1,z_2,z_3)$. Test $A_3(1,1,z)$.	A factor of the form $_4M_1(z)= z\{z-e^{-a_4T} f_7(b_4,c_4)\}$ exists.	$_4J_1(z)$ is a second-order linear system, $f_7(b_4,c_4)=\{\cos b_4T- c_4 \sin b_4T\}$ (A.5.2)
		A factor of the form $_4M_1(z)=(d_{41}ze^{-d_{42}T}-d_{43}e^{-d_{44}T})$, exists.	$_4J_1(z)$ is a first or second-order linear system and d_{41} to d_{44} are determined by comparison.
		No such factor exists.	$_4J_1(z)$ is either absent or a first or second-order system.
3.	Divide $A_3(z_1,z_2,z_3)$ by $_4M_1(z_3)$, if it	A common factor of the form $_3M_1(z)= z\{z-e^{-a_3T} f_8(b_3,c_3)\}$ exists.	$_3J_1(z)$ is a second-order linear system and $f_8(b_3,c_3)=(\cos b_3T- c_3 \sin b_3T)$ (A.5.3)

	exists in step 2, to give $B_2(z_1, z_2)$. Test $B_2(1,z)$ and $B_2(z,1)$ for a common factor.	A common factor of the form ${}_3M_1(z)=(d_{31}z\, e^{-d_{32}T}-d_{33}e^{-d_{34}T})$ exists.	${}_3J_1(z)$ is a first or second-order linear system and d_{31} to d_{34} are determined by comparison.
		No such common factor exists.	${}_3J_1(z)$ is either absent or a first or second-order system.
	Compute $B_2(z,z^{-1})$, if necessary.	${}_3M_1(z)$ is a constant.	${}_3J_1(z)$ is a first or second-order linear system.
4.	Divide $B_2(z_1,z_2)$ by ${}_3M_1(z_1z_2)$, if it exists in step 3, to give $D_2(z_1,z_2)$. Test $D_2(1,z)$ for a factor.	A factor of the form ${}_2M_1(z)= z\{z- e^{-a_2T} \times f_9(b_2,c_2)\}$, exists.	${}_2J_1(z)$ is a second-order linear linear system and $f_9(b_2,c_2)=(\cos b_2T- c_2\sin b_2T)$ (A.5.4)
		A factor of the form ${}_2M_1(z)=(d_{21}ze^{-d_{22}T}-d_{23}e^{-d_{24}T})$, exists.	${}_2J_1(z)$ is a first or second-order linear system and d_{21} to d_{24} are determined by comparison.
		No such factor exists.	${}_2J_1(z)$ is absent or a first or second-order system.
5.	Divide $D_2(z_1,z_2)$ by ${}_2M_1(z_2)$, if it exists in step 4, to give ${}_1M_1(z_1)$	${}_1M_1(z)= z\{z- e^{-a_1T} f_{10}(b_1,c_1)\}$, exists.	${}_1J_1(z)$ is a second-order system and $f_{10}(b_1,c_1)=(\cos b_1T- c_1\sin b_1T)$ (A.5.5)
		${}_1M_1(z)=(d_{11}ze^{-d_{12}T}-d_{13} \times e^{-d_{14}T})$ exists.	${}_1J_1(z)$ is a first or second-order linear system and d_{11} to d_{14} are determined by comparison.
		${}_1M_1(z)$ is a constant.	${}_1J_1(z)$ is absent or a first or second-order linear system.
6.	Compare $N_3(z,1,1)$, $N_3(1,z,1)$ and $N_3(1,1,z)$ for a common factor.	A common factor of the form ${}_5N_1(z)=(z^2- 2z\, e^{-a_5T}\cos b_5T+ e^{-2a_5T})$, exists in each of them.	${}_5J_1(m,z)$ is a second-order linear system. Knowing a_5 and b_5, c_5 can be determined from eqn.(A.5.1), and hence ${}_5J_1(m,z)$ can be realised.

		A common factor of the form ${}_5N_1(z)=(z-e^{-a_5T})$, exists in each of them.	${}_5J_1(m,z)$ is a first-order linear system. Knowing a_5, ${}_5J_1(m,z)$ can be realised.
		No such common factor exists in any of them.	${}_5J_1(m,z)$ is absent.
	Compute $N_3(1,z,z^{-1})$ and $N_3(z^2,z^{-1},z^{-1})$, if necessary.	${}_5N_1(z)$ is a constant.	
7.	Divide $N_3(z_1,z_2,z_3)$ by ${}_5N_1(z_1.z_2z_3)$, if it exists in step 6, to give $C_3(z_1,z_2,z_3)$. Compute $C_3(1,1,z)$.	A factor of the form ${}_4N_1(z)=(z^2-2ze^{-a_4T}\cos b_4T+e^{-2a_4T})$ exists.	${}_4J_1(z)$ is a second-order linear system. Knowing a_4 and b_4, c_4 can be calculated from eqn. (A.5.2) and hence ${}_4J_1(z)$ can be synthesised.
		A factor of the form ${}_4N_1(z)=(z-e^{-a_4T})$ exists.	${}_4J_1(z)$ is a first-order linear system. Knowing a_4, ${}_4J_1(z)$ can be realised.
		No such factor exists and $C_3(1,1,z)$ is a constant.	${}_4J_1(z)$ is absent.
8.	Divide $C_3(z_1,z_2,z_3)$ by ${}_4N_1(z_3)$, if it exists in step 7, to give $F_2(z_1,z_2)$. Compare $F_2(1,z)$ and $F_2(z,1)$ for a common factor.	A common factor ${}_3N_1(z)=(z^2-2ze^{-a_3T}\cos b_3T+e^{-2a_3T})$ exists.	${}_3J_1(z)$ is a second-order linear system. Knowing the values of a_3 and b_3, c_3 can be determined from eqn.(A.5.3) and hence ${}_3J_1(z)$ can be completely synthesised.
		A common factor of the form ${}_3N_1(z)=(z-e^{-a_3T})$, exists.	${}_3J_1(z)$ is a first-order linear system and can be realised, knowing the value of a_3.
		No such common factor exists.	${}_3J_1(z)$ is absent.
	Compute $F_2(z,z^{-1})$, if necessary.	${}_3N_1(z)$ is a constant.	

9.	Divide $F_2(z_1,z_2)$ by ${}_3N_1(z_1z_2)$, if it exists in step 8, to give $K_2(z_1,z_2)$. Test $K_2(1,z)$ for a factor.	A factor of the form ${}_2N_1(z)=(z^2-2z\,e^{-a_2T}\cos b_2T+e^{-2a_2T})$, exists.	${}_2J_1(z)$ is a second-order linear system. Knowing a_2 and b_2, c_2 can be determined from eqn.(A.5.4), and then ${}_2J_1(z)$ can be synthesised.
		A factor of the form ${}_2N_1(z)=(z-e^{-a_2T})$, exists.	${}_2J_1(z)$ is a first-order linear system, and knowing a_2, ${}_2J_1(z)$ can be realised.
		No such factor exists and $K_2(1,z)$ is a constant.	${}_2J_1(z)$ is absent.
10.	Divide $K_2(z_1,z_2)$ by ${}_2N_1(z_2)$, if it exists in step 9, to give ${}_1N_1(z_1)$.	${}_1N_1(z)=(z^2-2z\,e^{-a_1T}\cos b_1T+e^{-2a_1T})$, exists.	${}_1J_1(z)$ is a second-order linear system. Knowing a_1 and b_1, c_1 can be determined from eqn.(A.5.5), and hence ${}_1J_1(z)$ can be completely synthesised.
		${}_1N_1(z)=(z-e^{-a_1T})$, exists.	${}_1J_1(z)$ is a first-order linear system and can be realised knowing a_1.
		${}_1N_1(z)$ is a constant.	${}_1J_1(z)$ is absent.
11.	$\dfrac{M_3(m,z_1,z_2,z_3)}{N_3(z_1,z_2,z_3)}$	$k\,\dfrac{{}_5M_1(m,z_1z_2z_3)\,{}_4M_1(z_3)\,{}_3M_1(z_1z_2)\,{}_2M_1(z_2)\,{}_1M_1(z_1)}{{}_5N_1(z_1z_2z_3)\,{}_4N_1(z_3)\,{}_3N_1(z_1z_2)\,{}_2N_1(z_2)\,{}_1N_1(z_1)}$	Comparing both, the constsnt k can be determined.

APPENDIX A.7

DERIVATION OF DISCRETE STATE EQUATION TO CHARACTERISE MULTIVARIABLE SAMPLED-DATA NONLINEAR SYSTEMS

The solution to the state equation of the continuous part of the system shown in Fig.7.2, is given, in the Volterra series form, by

$$ {}_pX(s) = {}_pX_1(s)+{}_pX_2(s_1,s_2)+{}_pX_3(s_1,s_2,s_3)+\ldots\ldots. \tag{A.7.1}$$

where ${}_pX_1(s)$, ${}_pX_2(s_1,s_2)$, ${}_pX_3(s_1,s_2,s_3)$, are given by eqns.(6.3.4) to (6.3.6). The associated transforms ${}_pX_2(s)$ and ${}_pX_3(s)$ of ${}_pX_2(s_1,s_2)$ and ${}_pX_3(s_1,s_2,s_3)$, respectively, are obtained, using complex and real convolution theorems[85,86], as

$$ {}_pX_2(s) = {}_p\Phi^a_I(s)\Big[{}_IB^a_{ij}\ {}_iX_1(s) * {}_jX_1(s)+{}_IB^b_{iL}\ {}_iX_1(s) * {}_LZ_1(s) + {}_IB^c_{LM}\ {}_LZ_1(s) * {}_MZ_1(s)\Big] \tag{A.7.2}$$

and

$$ {}_pX_3(s) = {}_p\Phi^a_I(s)\Big[{}_IB^a_{ij}\{{}_iX_1(s) * {}_jX_2(s)+{}_iX_2(s) * {}_jX_1(s)\}+{}_IB^b_{iL}\ {}_iX_2(s) * {}_LZ_1(s) + {}_IC^a_{ijk}\ {}_iX_1(s) * {}_jX_1(s) * {}_kX_1(s)+{}_IC^b_{ijL}\ {}_iX_1(s) * {}_jX_1(s) * {}_LZ_1(s) + {}_IC^c_{iLM}\ {}_iX_1(s) * {}_LZ_1(s) * {}_MZ_1(s)+{}_IC^d_{LMN}\ {}_LZ_1(s) * {}_MZ_1(s) * {}_NZ_1(s)\Big] \tag{A.7.3}$$

Substituting ${}_pX_2(s)$ and ${}_pX_3(s)$ for ${}_pX_2(s_1,s_2)$ and ${}_pX_3(s_1,s_2,s_3)$, respectively, in eqn.(A.7.1) and inverting gives

$$ {}_px(t) = {}_px_1(t)+{}_px_2(t)+{}_px_3(t)+\ldots\ldots \tag{A.7.4}$$

where

$$ {}_px_1(t) = {}_p\phi^a_I(t)\,{}_Ix(0)+\int_0^t {}_p\phi^a_I(t-\tau)\,{}_IA^b_L\ {}_Lz(\tau)\ d\tau \tag{A.7.5}$$

$$ {}_px_2(t) = \int_0^t {}_p\phi^a_I(t-\tau)\{{}_IB^a_{ij}\ {}_ix_1(\tau)\,{}_jx_1(\tau)+{}_IB^b_{iL}\ {}_ix_1(\tau)\,{}_Lz(\tau)+{}_IB^c_{LM}\ {}_Lz(\tau)\,{}_Mz(\tau)\}\,d\tau \tag{A.7.6}$$

and

$$ {}_px_3(t) = \int_0^t {}_p\phi^a_I(t-\tau)\Big[{}_IB^a_{ij}\{{}_ix_1(\tau)\,{}_jx_2(\tau)+{}_ix_2(\tau)\,{}_jx_1(\tau)\}+{}_IB^b_{iL}\ {}_ix_2(\tau)\,{}_Lz(\tau) + {}_IC^a_{ijk}\ {}_ix_1(\tau)\,{}_jx_1(\tau)\,{}_kx_1(\tau)+{}_IC^b_{ijL}\ {}_ix_1(\tau)\,{}_jx_1(\tau)\,{}_Lz(\tau) + {}_IC^c_{iLM}\ {}_ix_1(\tau)\,{}_Lz(\tau)\,{}_Mz(\tau)+{}_IC^d_{LMN}\ {}_Lz(\tau)\,{}_Mz(\tau)\,{}_Nz(\tau)\Big]\ d\tau \tag{A.7.7}$$

where $\phi^a(t) = L^{-1}\{\Phi^a(s)\}$, is the state transition matrix of the continuous system. Since the signals ${}_Lz(t)$, $L= 1,2,\ldots,R$ are the outputs of the zero-order hold device, they are described by

$$ {}_Lz(t) = {}_Lu(KT) \;;\quad KT \leq t \leq (K+1)T \tag{A.7.8}$$

where $K = 0,1,2,\ldots$ etc., and ${}_Lz(\tau) = {}_Lu(KT)$ is a constant vector over the sampling interval. The eqns.(A.7.4) to (A.7.7) are useful only when the initial time is taken at $t=0$. However, in the study of digital control systems, it is desirable to use a more general time reference, t_0. Thus, taking $t=t_0$ as the initial time and making use of the fact that ${}_Lz(\tau) = {}_Lu(KT)$ is a constant vector, eqns.(A.7.5) to (A.7.7), become

$$ {}_px_1(t) = \{{}_p\phi^a_I(t-t_0)\,{}_Ix(t_0) + {}_I\phi^b_L(t-t_0)\,{}_Lu(KT)\} \tag{A.7.9}$$

$$\begin{aligned} {}_px_2(t) = \{&{}_p\theta^a_{ij}(t-t_0)\,{}_ix(t_0)\,{}_jx(t_0) + {}_p\theta^b_{iL}(t-t_0)\,{}_ix(t_0)\,{}_Lu(KT) \\ &+ {}_p\theta^c_{LM}(t-t_0)\,{}_Lu(KT)\,{}_Mu(KT)\} \end{aligned} \tag{A.7.10}$$

$$\begin{aligned} {}_px_3(t) = \{&{}_p\psi^a_{ijk}(t-t_0)\,{}_ix(t_0)\,{}_jx(t_0)\,{}_kx(t_0) + {}_p\psi^b_{ijL}(t-t_0)\,{}_ix(t_0)\,{}_jx(t_0)\,{}_Lu(KT) \\ &+ {}_p\psi^c_{iLM}(t-t_0)\,{}_ix(t_0)\,{}_Lu(KT)\,{}_Mu(KT) + {}_p\psi^d_{LMN}(t-t_0)\,{}_Lu(KT)\,{}_Mu(KT)\,{}_Nu(KT)\} \end{aligned} \tag{A.7.11}$$

where

$$ {}_p\phi^b_L(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_J(t-\tau)\; {}_JA^b_L \, d\tau $$

$$ {}_p\theta^a_{ij}(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_I(t-\tau)\{{}_IB^a_{JK}\; {}_J\phi^a_i(\tau)\,{}_K\phi^a_j(\tau)\}\, d\tau $$

$$ {}_p\theta^b_{iL}(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_I(t-\tau)\left[{}_IB^a_{JK}\{{}_J\phi^a_i(\tau)\,{}_K\phi^b_L(\tau) + {}_J\phi^b_L(\tau)\,{}_K\phi^a_i(\tau)\} + {}_IB^b_{jL}\; {}_j\phi^a_i(\tau)\right] d\tau $$

$$ {}_p\theta^c_{LM}(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_I(t-\tau)\{{}_IB^a_{JK}\; {}_J\phi^b_L(\tau)\,{}_K\phi^b_M(\tau) + {}_IB^b_{JL}\; {}_J\phi^b_M(\tau) + {}_IB^c_{LM}\}\, d\tau $$

$$\begin{aligned} {}_p\psi^b_{ijL}(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_I(t-\tau)\Big[&{}_IB^a_{ST}\{{}_S\phi^a_i(\tau)\,{}_T\theta^b_{jL}(\tau) + {}_S\phi^b_L(\tau)\,{}_T\theta^a_{ij}(\tau) + {}_S\theta^a_{ij}(\tau)\,{}_T\phi^b_L(\tau) \\ &+ {}_S\theta^b_{iL}(\tau)\,{}_T\phi^a_j(\tau)\} + {}_IC^a_{STZ}\{{}_S\phi^a_i(\tau)\,{}_T\phi^a_j(\tau)\,{}_Z\phi^b_L(\tau) \\ &+ {}_S\phi^a_i(\tau)\,{}_Z\phi^a_j(\tau)\,{}_T\phi^b_L(\tau) + {}_S\phi^b_L(\tau)\,{}_T\phi^a_i(\tau)\,{}_Z\phi^a_j(\tau)\} + {}_IB^b_{SL}\; {}_S\theta^a_{ij}(\tau) \\ &+ {}_IC^b_{STL}\; {}_S\phi^a_i(\tau)\,{}_T\phi^a_j(\tau)\Big] \, d\tau \quad , \end{aligned}$$

$$ {}_p\psi^a_{ijk}(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_I(t-\tau)\Big[{}_IB^a_{ST}\{{}_S\phi^a_i(\tau)_T\theta^a_{jk}(\tau)+{}_S\theta^a_{ij}(\tau)_T\phi^a_k(\tau)\} + {}_IC^a_{STZ}\ {}_S\phi^a_i(\tau)_T\phi^a_j(\tau)_Z\phi^a_k(\tau)\Big]\ d\tau\ , $$

$$ {}_p\psi^c_{iLM}(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_I(t-\tau)\Big[{}_IB^a_{ST}\{{}_S\phi^a_i(\tau)_T\theta^c_{LM}(\tau)+{}_S\theta^b_{iL}(\tau)_T\phi^b_M(\tau)+{}_S\phi^b_L(\tau)_T\theta^b_{iM}(\tau) + {}_S\theta^c_{LM}(\tau)_T\phi^a_i(\tau)\}+{}_IC^a_{STZ}\{{}_S\phi^a_i(\tau)_T\phi^b_L(\tau)_Z\phi^b_M(\tau) + {}_S\phi^b_L(\tau)_T\phi^a_i(\tau)_Z\phi^b_M(\tau)+{}_S\phi^b_L(\tau)_T\phi^b_M(\tau)_Z\phi^a_i(\tau)\}+{}_IC^b_{STL}\{{}_S\phi^a_i(\tau)_T\phi^b_M(\tau) + {}_S\phi^b_M(\tau)_T\phi^a_i(\tau)\}+{}_IB^b_{SM}\ {}_S\theta^b_{iL}(\tau) + {}_IC^c_{SLM}\ {}_S\phi^a_i(\tau)\Big]\ d\tau\ , $$

$$ {}_p\psi^d_{LMN}(t-t_0) = \int_{t_0}^{t} {}_p\phi^a_I(t-\tau)\Big[{}_IB^a_{ST}\{{}_S\phi^b_L(\tau)_T\theta^c_{MN}(\tau)+{}_S\theta^c_{LM}(\tau)_T\phi^b_N(\tau)\}+{}_IB^b_{SN}\ {}_S\theta^c_{LM}(\tau) + {}_IC^a_{STZ}\ {}_S\phi^b_L(\tau)_T\phi^b_M(\tau)_Z\phi^b_N(\tau)+{}_IC^b_{STN}\ {}_S\phi^b_L(\tau)_T\phi^b_M(\tau)+{}_IC^c_{SMN}\ {}_S\phi^b_L(\tau) + {}_IC^d_{LMN}\Big]\ d\tau $$

$$ \text{(A.7.12)} $$

It is to be noted that eqn.(A.7.10) is derived by using eqn.(A.7.9) in eqn.(A.7.6) and eqn.(A.7.11) is derived by using eqns.(A.7.9) and (A.7.10) in eqn.(A.7.7). Substituting eqns.(A.7.9) to (A.7.11) into eqn. (A.7.4) and letting t_0=KT and t=(K+1)T, gives the discrete state equation (7.5.2) of the system shown in Fig.7.2.